PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Continuous Absorption

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V. *Continuous Absorption.*

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(*Communicated by* R. H. Fowler, *F.R.S.*)

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§ 1. *Introduction.*

For some years astrophysicists have been looking for an adequate theory of continuous—as opposed to line—absorption. The natural and generally accepted mechanism is the transition of an electron from a bound state to a free state, or from one free state in the neighbourhood of an ion to another free state of greater energy. The theory hitherto used is Kramers' theory† of the converse process of emission by a free electron passing a positive nucleus. Since emission and absorption are intimately connected by thermodynamics, the absorption coefficient can be calculated from Kramers' formulæ.‡ Unfortunately, although Kramers' work is in good agreement with laboratory observations of X-rays, it gives an absorption coefficient many times smaller than that found from astronomical observations.

Kramers used classical electromagnetism, and got over the difficulty of the quantisation of negative energies by distributing the classical emission that involved captures somewhat arbitrarily among the various stationary states. It was evidently desirable to do the same work by means of quantum theory, both for the sake of greater rigour, and in the hope of finding a larger absorption. The foundations of such a theory were laid by Oppenheimer,|| upon the bed-rock of Schrödinger's equation, in a paper to which this one is much indebted. The matrix-elements involving positive energies present considerable difficulty, and the approximations used by Oppenheimer in his paper of 1927 are unsuitable for stellar applications.

The present work carries the theory further, though not to completion. It contemplates absorption by an electron in an encounter with a positive nucleus, or by an electron bound to the nucleus. Interference by other electrons is not considered. In the

* As Mr. Gaunt has left Europe, this paper has been seen through the press by Dr. Dirac and Mr. R. H. Fowler. They trust they have done justice to the author's work. Any communications on the subject matter covered by this paper should be addressed to one or other of them.

† Kramers, 'Phil. Mag.,' vol. 46, p. 836 (1923).

‡ Eddington, 'Internal Constitution of the Stars,' p. 229 (1926): Milne, 'M.N.R.A.S.,' vol. 85, p. 750 (1925). Referred to as M I.

|| Oppenheimer, 'Z. Physik,' vol. 41, p. 768 (1927). Referred to as O I.

 [Published May 13, 1930.

interior of a star, where the atoms are highly ionised, this model should be a good one. Relativity effects and quadripole radiation are neglected. Even with these simplifications the problem is not too easy, and it has been solved only in certain limiting cases.

These calculations were far advanced when another paper was published by OPPENHEIMER* covering the same ground. By using parabolic instead of polar co-ordinates, he gains somewhat in elegance, and is able to distinguish the polarisation of the radiation. He works throughout in terms of emission instead of absorption, but this difference is trivial. The striking feature of his paper is the large absorption coefficient, approaching the astrophysical value. An obvious discrepancy between some of our results provoked a careful checking of OPPENHEIMER'S paper. The outcome was the discovery of a mistake whose correction brings his results into line with ours, but unfortunately spoils his absorption coefficient.†

The plan of this paper is as follows. § 2 recapitulates the properties of the wave-functions, especially those for positive energies, in a convenient form. § 3 proceeds to the calculation of the matrix-elements which are required in the theory of emission and absorption. The usual integrals giving the matrix-elements of the electric moment are divergent when positive energies are concerned, and therefore the matrix-elements of the electron's acceleration are used instead. The contents of subsection 3.1 are well known, but the rest of the section breaks new ground. The resulting expressions for the matrix-elements contain integrals to which various approximations are made in § 4. § 5 begins with a brief restatement of absorption theory suitable to the continuous spectrum, and gives absorption coefficients in the various limiting cases contemplated in § 4. In § 6 comparisons are made with the work of KRAMERS, MILNE, and OPPENHEIMER, and stellar applications are discussed. The conclusions are summarised in § 7.

An abridged version of this paper has been published elsewhere.‡ Another paper on this subject has recently been published by SUGUIRA,|| but it has not been possible to correlate his results with those of this paper.

§ 2. *Wave-Functions.*¶

2.0. We start with SCHRÖDINGER'S equation for an electron in the field of a central charge Ze. In the usual notation,

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{Ze^2}{r}\right)\psi = 0. \qquad (2.00)$$

* OPPENHEIMER, 'Z. Physik,' vol. 55, p. 725 (1929). Referred to as O II.

† GAUNT, 'Z. Physik,' vol. 59. p. 508 (1930).

‡ GAUNT, 'Roy. Soc. Proc.,' A, vol. 126, p. 654 (1930).

|| SUGUIRA, 'Phys. Rev.,' vol. 34, p. 858 (1929).

¶ *Cf.* SCHRÖDINGER, 'Ann. Physik,' vol. 79, p. 734 (1926); OPPENHEIMER, 'Z. Physik,' vol. 41, p. 26 (1927).

We write the wave-function for any stationary state in the form

$$\psi(\mathrm{E}, k, u) = \xi_k^u \mathrm{P}_k^u(\theta, \phi)\, \xi(\mathrm{E}, k)\, u(\mathrm{E}, k; r), \quad \ldots \quad (2.01)$$

where (following DARWIN)

$$\mathrm{P}_k^u(\theta, \phi) = (k-u)!\ \sin^u\theta \left(\frac{d}{d\cos\theta}\right)^{k+u} \frac{(\cos^2\theta - 1)^k}{2^k k!}\, e^{iu\phi}. \quad \ldots \quad (2.02)$$

and ξ_k^u is its normalising factor, given by

$$(\xi_k^u)^{-2} = \frac{4\pi}{2k+1}(k-u)!\ (k+u)! \quad \ldots \quad (2.03)$$

The radial part of the wave-function satisfies the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \left(\mp\gamma^2 + \frac{\alpha}{r} - \frac{k(k+1)}{r^2}\right)u = 0, \quad \ldots \quad (2.04)$$

where we have written

$$\gamma = \frac{2\pi}{h}\sqrt{2m\,|\mathrm{E}|}, \qquad \alpha = \frac{8\pi^2 m Z e^2}{h^2}, \quad \ldots \quad (2.05)$$

and the upper sign corresponds to negative, the lower to positive, energies. We shall also use

$$n = \alpha/2\gamma, \qquad \mathrm{E} = \mp\frac{2\pi^2 m Z^2 e^4}{h^2 n^2} \quad \ldots \quad (2.06)$$

2.1. For negative energies n must be an integer greater than k and

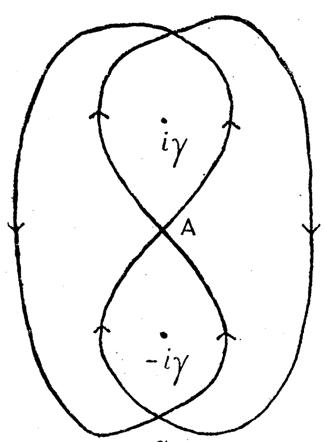
$$u(\mathrm{E}, k; r) = r^k e^{-\gamma r} \mathop{\mathrm{L}}_{n+k}^{2k+1}(2\gamma r), \quad \ldots \quad (2.10)$$

$$= -r^k e^{-\gamma r} \sum_{q=0}^{n-k-1} \frac{(n+k)!^2\,(-2\gamma r)^q}{q!\,(2k+1+q)!\,(n-k-1-q)!}. \quad (2.11)$$

Positive energies are unrestricted, and so is the corresponding n. We may take as the wave-function

$$u = r^k \int_{\mathrm{C}} e^{zr}(z - i\gamma)^{k-in}(z + i\gamma)^{k+in}\, dz, \quad \ldots \quad (2.12)$$

where C is a double curve encircling $\pm i\gamma$ in the manner shown in the diagram, and the integrand is determined by making $|\arg(z \pm i\gamma)| < \pi$ at the point A.



C.
FIG. 1.

Some recurrence formulæ can be proved at once. Using the series (2.11) we find

$$2(k+1)\left(\frac{d}{dr}-\frac{k}{r}\right)u(\mathrm{E},k)+\alpha u(\mathrm{E},k)=\frac{-\alpha^2}{n^2(n+k+1)}u(\mathrm{E},k+1)\quad(\mathrm{E}<0),\quad(2.13)$$

$$2k\left(\frac{d}{dr}+\frac{k+1}{r}\right)u(\mathrm{E},k)-\alpha u(\mathrm{E},k)=(n+k)^2(n-k)\,u(\mathrm{E},k-1)\quad(\mathrm{E}<0).\quad(2.14)$$

Differentiating (2.12) under the integral sign and integrating by parts we find

$$2(k+1)\left(\frac{d}{dr}-\frac{k}{r}\right)u(\mathrm{E},k)+\alpha u(\mathrm{E},k)=-u(\mathrm{E},k+1)\qquad(\mathrm{E}>0),\quad(2.15)$$

$$2k\left(\frac{d}{dr}+\frac{k+1}{r}\right)u(\mathrm{E},k)-\alpha u(\mathrm{E},k)=\frac{\alpha^2}{n^2}(n^2+k^2)\,u(\mathrm{E},k-1)\quad(\mathrm{E}>0).\quad(2.16)$$

2.2. We require the asymptotic behaviour of (2.12) as $r\to\infty$. Let C_1 be a contour starting at $i\gamma-\infty$, encircling $i\gamma$ positively, and returning to $i\gamma-\infty$; C_2 a similar contour for $-i\gamma$.

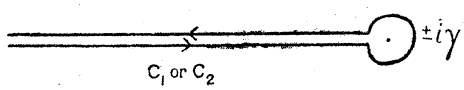


FIG. 2.

Let u_1, u_2 be the functions obtained when C_1, C_2 are substituted for C in (2.12), and the integrand is determined by $|\arg(z\pm i\gamma)|\leq\pi$. The integrand is bounded near $\pm i\gamma$. Putting $z=\pm i\gamma-x$,

$$\left.\begin{aligned}u_1&=(-)^{k+1}(e^{\pi n}-e^{-\pi n})\,r^k\int_0^\infty e^{i\gamma r-xr}x^{k-in}(2i\gamma-x)^{k+in}\,dx\\ u_2&=(-)^{k}(e^{\pi n}-e^{-\pi n})\,r^k\int_0^\infty e^{-i\gamma r-xr}x^{k+in}(-2i\gamma-x)^{k-in}\,dx\end{aligned}\right\},\quad.\ .\quad(2.20)$$

where

$$\arg x=0,\qquad|\arg(\pm 2i\gamma-x)|<\pi.$$

Evidently iu_1 and iu_2 are conjugate. As $r\to\infty$, the important parts of the integrals in (2.20) are those for which x is small. Expanding $(2i\gamma-x)^{k+in}$ in powers of x we obtain

$$u_1\sim(-)^{k+1}i^k e^{\frac{1}{2}\pi n}(1-e^{-2\pi n})(2\gamma)^{k+in}\,\Gamma(k+1-in)\,e^{i\gamma r}r^{-1+in}$$
$$\times\left(1-\frac{(k+in)(k+1-in)}{2i\gamma r}+\ldots\right).\quad.\ .\quad(2.21)$$

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By deforming C as is shown here, we find u in terms of u_1 and u_2

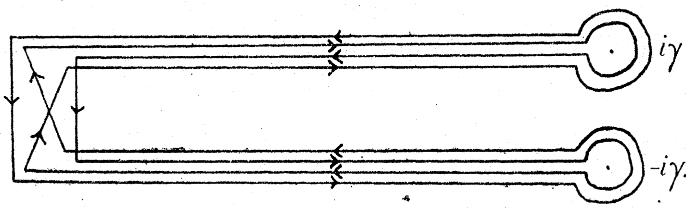

FIG. 3.

$$u = (1 - e^{-2\pi n})(u_1 + u_2). \quad \quad (2.22)$$

Thus as $r \to \infty$

$$u \sim - 2i\mathfrak{R}(-i)^{k+1} e^{\frac{1}{2}\pi n} (1 - e^{-2\pi n})^2 (2\gamma)^{k+in} \Gamma(k + 1 - in) e^{i\gamma r} r^{-1+in}$$
$$\times \left(1 - \frac{(k + in)(k + 1 - in)}{2i\gamma r} + \ldots\right). \quad \quad (2.23)$$

The leading term of (2.23) may be written

$$u \sim - \frac{2i\mathrm{K}}{r} \sin(\gamma r + n \log r + \beta), \quad \quad (2.24)$$

where

$$\mathrm{K} = e^{\frac{1}{2}\pi n} (1 - e^{-2\pi n})^2 (2\gamma)^k \,|\, \Gamma(k + 1 - in)|, \quad \quad (2.25)$$

and β is independent of r.

2.3. It is well known that the functions (2.11) are orthogonal to each other and to the functions (2.12). The corresponding property for the functions (2.12) may be expressed in the form

$$\lim_{\epsilon \to 0} \int_0^\infty u(\mathrm{E}, k)\, u(\mathrm{E}', k)\, e^{-\epsilon r} r^2\, dr = 0 \qquad (\mathrm{E} \neq \mathrm{E}'). \quad \quad (2.30)$$

The exponential factor is necessary to secure convergence, as is shown by (2.24). The proof of (2.30), which follows, is of the usual form.

We write (2.04) in the form

$$\frac{d^2}{dr^2}(ru) + \left(+ \gamma^2 + \frac{\alpha}{r} - \frac{k(k+1)}{r}\right) ru = 0, \quad \quad (2.31)$$

so that

$$\gamma^2 \lim_{\epsilon \to 0} \int_0^\infty ru(\mathrm{E}, k)\, ru(\mathrm{E}', k)\, e^{-\epsilon r}\, dr$$
$$= - \lim_{\epsilon \to 0} \int_0^\infty \left(\frac{d^2}{dr^2} + \frac{\alpha}{r} - \frac{k(k+1)}{r^2}\right) r\, u(\mathrm{E}, k) \,.\, r\, u(\mathrm{E}', k)\, e^{-\epsilon r}\, dr,$$
$$= - \lim_{\epsilon \to 0} \int_0^\infty r\, u(\mathrm{E}, k) \left(\frac{d}{dr^2} + \frac{\alpha}{r} - \frac{k(k+1)}{r^2}\right) r\, u(\mathrm{E}', k)\, e^{-\epsilon r}\, dr,$$

after two integrations by parts. The exponential term can be placed before the differential operator without affecting the limit. The result is the left-hand side with γ' (corresponding to E′) for γ. (2.30) follows.

2.4. It remains to find the normalising factor ξ (E, k). For negative energies it is given as usual by

$$\int_0^\infty \xi(\mathrm{E}, k)^2 \, |u(\mathrm{E}, k; r)|^2 r^2 \, dr = 1, \quad \ldots \quad (2.40)$$

whence

$$\xi(\mathrm{E}, k) = (2\gamma)^{k+\frac{3}{2}} \left[\frac{(n-k-1)!}{(n+k)!^3 \, 2n}\right]^{\frac{1}{2}}. \quad \ldots \quad (2.41)$$

The wave-functions for positive energies are normalised as follows, after Oppenheimer. If

$$\Delta\Theta \equiv \int_{\mathrm{E}}^{\mathrm{E}+\Delta\mathrm{E}} \xi(\mathrm{E}, k) \, u(\mathrm{E}, k) \, d\mathrm{E}/h, \quad \ldots \quad (2.42)$$

and $\mathrm{E} < \mathrm{E}' < \mathrm{E} + \Delta\mathrm{E}$, then

$$\int_0^\infty \xi(\mathrm{E}', k) \, u^*(\mathrm{E}', k) \, \Delta\Theta r^2 \, dr \to 1 \text{ as } \Delta\mathrm{E} \to 0. \quad \ldots \quad (2.43)$$

With this normalisation, under certain conditions for F (r),

$$\mathrm{F}(r) \, \xi(\mathrm{E}', k) \, u(\mathrm{E}', k) = \sum_{\mathrm{E}<0} (\mathrm{E}, k \,|\mathrm{F}|\, \mathrm{E}', k) \, \xi(\mathrm{E}, k) \, u(\mathrm{E}, k)$$
$$+ \int_0^\infty (\mathrm{E}, k \,|\mathrm{F}|\, \mathrm{E}', k) \, \xi(\mathrm{E}, k) \, u(\mathrm{E}, k) \, d\mathrm{E}/h, \quad \ldots \quad (2.44)$$

where

$$(\mathrm{E}, k \,|\mathrm{F}|\, \mathrm{E}', k) = \left\{\begin{array}{ll} \int_0^\infty \xi(\mathrm{E}, k) \, u^*(\mathrm{E}, k) \, \mathrm{F} \, \xi(\mathrm{E}', k) \, u(\mathrm{E}', k) \, r^2 \, dr & (\mathrm{E} < 0) \\ \lim_{\Delta\mathrm{E}\to 0} \dfrac{h}{\Delta\mathrm{E}} \int_0^\infty \Delta\Theta^* \, \mathrm{F} \, \xi(\mathrm{E}', k) \, u(\mathrm{E}', k) \, r^2 \, dr & (\mathrm{E} > 0) \end{array}\right\}, \quad . \quad (2.45)$$

For any finite range of r the integral (2.43) vanishes when $\Delta\mathrm{E} \to 0$. Thus we need consider only large r. By (2.24)

$$\Delta\Theta \sim \xi \int_\gamma^{\gamma+\Delta\gamma} - \frac{2i\mathrm{K}}{r} \sin\left(\gamma r + \frac{\alpha}{2\gamma} \log r + \beta\right) \frac{h}{4\pi^2 m} \gamma \, d\gamma,$$

$$\sim - \xi \frac{2i\mathrm{K}}{r^2} \frac{h\gamma}{4\pi^2 m} \left[\cos\left(\gamma r + \frac{\alpha}{2\gamma} \log r + \beta\right)\right.$$
$$\left. - \cos\left(\gamma r + \frac{\alpha}{2\gamma} \log r + \beta + \left\{r - \frac{\alpha}{2\gamma^2} \log r\right\} \Delta\gamma\right)\right],$$

for the variations of ξ and β with γ are negligible in the limit. If E′ corresponds to $\gamma + \Delta'\gamma$, so that $0 < \Delta'\gamma < \Delta\gamma$,

$$\xi u^* (\mathrm{E}', k)\, \Delta\Theta \sim \xi^2 \frac{4\mathrm{K}^2}{r^3} \frac{h\gamma}{8\pi^2 m} \Big[\sin\Big\{2\Big(\gamma r + \frac{\alpha}{2\gamma}\log r + \beta\Big) + \Big(r - \frac{\alpha}{2\gamma^2}\log r\Big)\Delta'\gamma\Big\}$$

$$+ \sin\Big(r - \frac{\alpha}{2\gamma^2}\log r\Big)\Delta'\gamma - \sin\Big\{2\Big(\gamma r + \frac{\alpha}{2\gamma}\log r + \beta\Big) + \Big(r - \frac{\alpha}{2\gamma^2}\log r\Big)(\Delta\gamma + \Delta'\gamma)\Big\}$$

$$+ \sin\Big(r - \frac{\alpha}{2\gamma^2}\log r\Big)(\Delta\gamma - \Delta'\gamma)\Big],$$

and (2.43) becomes in the limit

$$\xi^2 \cdot \frac{\mathrm{K}^2 h\gamma}{2\pi^2 m}\, 2 \int_0^\infty \frac{\sin r}{r}\, dr = 1.$$

Thus

$$\xi(\mathrm{E}, k) = \frac{1}{\mathrm{K}} \sqrt{\left(\frac{2\pi m}{h\gamma}\right)} = \frac{\sqrt{(4\pi m/h)}}{e^{\frac{1}{2}\pi n}(1 - e^{-2\pi n})^{\frac{1}{2}} (2\gamma)^{k+\frac{1}{2}} \,|\, \Gamma(k + 1 - in)\,|}. \qquad (2.46)$$

The meaning of (2.40) is that the wave-function for negative energy represents one electron. The meaning of the normalisation for positive energy can be found by combining (2.24) and (2.46). As $r \to \infty$

$$\xi(\mathrm{E}, k)\, u(\mathrm{E}, k; r) \sim -\frac{2i}{r} \sqrt{\left(\frac{2\pi m}{h\gamma}\right)} \sin(\gamma r + n \log r + \beta). \quad . \quad . \quad . \ (2.47)$$

This expression may be broken up into two parts representing outgoing and incoming streams of electrons. The density of the electrons in each of the streams represented by the complete wave-function (2.01) is

$$\frac{2\pi m}{h\gamma r^2}\, \xi_k^{u2}\, |\, \mathrm{P}_k^u \,|^2.$$

The velocity at infinity is

$$v = \sqrt{\frac{2\mathrm{E}}{m}} = \frac{h\gamma}{2\pi m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (2.48)$$

by (2.05), and this velocity is practically radial at large distances. Multiplying the density by v and integrating over a large sphere, we find that one electron enters or leaves the sphere in unit time. In other words, there is one encounter per unit time.

§ 3. *The Matrix-Elements of the Acceleration.*

3.0. The acceleration of the electron is proportional to $\mathbf{j}/r^2$, where $\mathbf{j}$ is the unit vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The matrix-elements are defined by the usual integrals

$$(\mathrm{E}, k, u \,|\, \mathbf{j}/r^2 \,|\, \mathrm{E}', k', u') = \int \psi^*(\mathrm{E}, k, u)\, \frac{\mathbf{j}}{r^2}\, \psi(\mathrm{E}', k', u')\, d\tau, \quad . \quad . \quad . \ (3.00)$$

the integration being over all space. Limits such as that in (2.45B) are unnecessary, as the integrals are strongly convergent; in fact (2.45B) reduces to the same form as (2.45A). The integrations with respect to θ, ϕ, and r, are all separate, and

$$(\mathrm{E}, k, u \,|\, \mathbf{j}/r^2 \,|\, \mathrm{E}', k', u') = (k, u \,|\, \mathbf{j} \,|\, k', u')\, (\mathrm{E}, k \,|\, 1/r^2 \,|\, \mathrm{E}', k') \quad . \quad . \quad . \ (3.01)$$

where

$$(k, u \,|\, \mathbf{j} \,|\, k', u') = \int_0^{\pi}\int_0^{2\pi} \xi_k^u \, \mathrm{P}_k^{u*} \, \mathbf{j} \, \xi_{k'}^{u'} \, \mathrm{P}_{k'}^{u'} \sin\theta \, d\theta \, d\phi. \quad . \quad . \quad . \quad . \quad . \quad (3.02)$$

The matrix-elements vanish unless $k' = k \pm 1$, $u' = u$ or $u \pm 1$. Also we are concerned only with cases in which at least one of the energies is positive. We shall suppose that E′ is positive and greater than E.

3.1. Using the formulæ

$$\left.\begin{aligned}
\int_0^{\pi}\int_0^{2\pi} \mathrm{P}_k^{u*} \cos\theta \, \mathrm{P}_{k+1}^{u} \sin\theta \, d\theta \, d\phi &= \frac{4\pi\,(k+u+1)!\,(k-u+1)!}{(2k+1)\,(2k+3)} \\
\int_0^{\pi}\int_0^{2\pi} \mathrm{P}_k^{u*} e^{-i\theta} \sin\theta \, \mathrm{P}_{k+1}^{u+1} \sin\theta \, d\theta \, d\phi &= \frac{4\pi\,(k+u+2)!\,(k-u)!}{(2k+1)\,(2k+3)} \\
\int_0^{\pi}\int_0^{2\pi} \mathrm{P}_k^{u*} e^{i\theta} \sin\theta \, \mathrm{P}_{k+1}^{u-1} \sin\theta \, d\theta \, d\phi &= -\frac{4\pi\,(k+u)!\,(k-u+2)!}{(2k+1)\,(2k+3)}
\end{aligned}\right\}, \quad (3.10)$$

and introducing the normalising factors ξ_k^u, etc., from (2.03), we find

$$\left.\begin{aligned}
(k, u \,|\, \cos\theta \,|\, k+1, u) &= \sqrt{\frac{(k+u+1)\,(k-u+1)}{(2k+1)\,(2k+3)}} \\
(k, u \,|\, e^{-i\phi}\sin\theta \,|\, k+1, u+1) &= \sqrt{\frac{(k+u+2)\,(k+u+1)}{(2k+1)\,(2k+3)}} \\
(k, u \,|\, e^{i\phi}\sin\theta \,|\, k+1, u-1) &= -\sqrt{\frac{(k-u+2)\,(k-u+1)}{(2k+1)\,(2k+3)}}
\end{aligned}\right\}, \quad (3.11)$$

Other choices of u' give integrals which vanish. We shall require squares of the matrix-elements. Denoting a scalar product by a dot, we have

$$\left.\begin{aligned}
(k, u \,|\, \mathbf{j} \,|\, k+1, u) \,.\, (k, u \,|\, \mathbf{j} \,|\, k+1, u)^* &= (k, u \,|\, \cos\theta \,|\, k+1, u)^2 \\
&= \frac{(k+u+1)\,(k-u+1)}{(2k+1)\,(2k+3)} \\
(k, u \,|\, \mathbf{j} \,|\, k+1, u+1) \,.\, (k, u \,|\, \mathbf{j} \,|\, k+1, u+1)^* &= 2\,(k, u \,|\, \tfrac{1}{2} e^{-i\phi}\sin\theta \,|\, k+1, u+1)^2 \\
&= \tfrac{1}{2}\,\frac{(k+u+2)\,(k+u+1)}{(2k+1)\,(2k+3)} \\
(k, u \,|\, \mathbf{j} \,|\, k+1, u-1) \,.\, (k, u \,|\, \mathbf{j} \,|\, k+1, u-1)^* &= 2\,(k, u \,|\, \tfrac{1}{2} e^{i\phi}\sin\theta \,|\, k+1, u-1)^2 \\
&= \tfrac{1}{2}\,\frac{(k-u+2)\,(k-u+1)}{(2k+1)\,(2k+3)}
\end{aligned}\right\}, \quad (3.12)$$

so that

$$\sum_{u'} (k, u \,|\, \mathbf{j} \,|\, k+1, u') \,.\, (k, u \,|\, \mathbf{j} \,|\, k+1, u')^* = \frac{k+1}{2k+1}. \quad . \quad . \quad . \quad . \quad (3.13)$$

Similarly

$$\left.\begin{aligned}(k, u \,|\, \mathbf{j} \,|\, k-1, u) \,.\, (k, u \,|\, \mathbf{j} \,|\, k-1, u)^* &= \frac{(k+u)\,(k-u)}{(2k-1)\,(2k+1)} \\ (k, u \,|\, \mathbf{j} \,|\, k-1, u+1) \,.\, (k, u \,|\, \mathbf{j} \,|\, k-1, u+1)^* &= \frac{(k-u)\,(k-u-1)}{(2k-1)\,(2k+1)} \\ (k, u \,|\, \mathbf{j} \,|\, k-1, u-1) \,.\, (k, u \,|\, \mathbf{j} \,|\, k-1, u-1)^* &= \frac{(k+u)\,(k+u-1)}{(2k-1)\,(2k+1)}\end{aligned}\right\}, \qquad (3.14)$$

so that

$$\sum_{u'} (k, u \,|\, \mathbf{j} \,|\, k \quad 1, u') \,.\, (k, u \,|\, \mathbf{j} \,|\, k-1, u')^* = \frac{k}{2k+1}. \quad . \quad . \quad . \quad (3.15)$$

These are the ordinary formulæ for the relative intensities of lines in the Zeemann effect.

3.2. We have now to evaluate $(\mathrm{E}, k \,|\, 1/r^2 \,|\, \mathrm{E}', k \pm 1)$. First let E be positive, and take the upper sign. By (2.20), (2.22), the wave-function is a pure imaginary; also the integrand behaves like $1/r^2$ as $r \to \infty$. Hence

$$\int_0^\infty u^* (\mathrm{E}, k\,;\; r) \frac{1}{r^2} u (\mathrm{E}', k+1\,;\; r)\, r^2\, dr$$

$$= -\lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon r} u (\mathrm{E}, k\,;\; r)\, u (\mathrm{E}', k+1\,;\; r)\, dr,$$

$$= -\lim_{\epsilon \to 0} \int_0^\infty dr \int_{\mathrm{C}} dz \int_{\mathrm{C}'} dz'\, e^{-(\epsilon - z - z')r}\, r^{2k+1}\, (z + i\gamma)^{k+in}\, (z - i\gamma)^{k-in}$$

$$(z' + i\gamma')^{k+1+in'}\, (z' - i\gamma')^{k+1-in'}, \quad . \quad . \quad . \quad . \quad . \quad (3.20)$$

using (2.12). The exponential factor is inserted in order to facilitate the next step, which is the inversion of the order of integration. This is permissible provided we choose C and C′ so that on them the real parts of z and z' are less than $\frac{1}{2}\varepsilon$. The integration with respect to r is then uniformly convergent for points on C and C′, and (3.20) becomes

$$-\int_{\mathrm{C}} dz \int_{\mathrm{C}'} dz' \frac{(2k+1)\,!}{(\varepsilon - z - z')^{2k+2}} (z + i\gamma)^{k+in}\, (z - i\gamma)^{k-in}\, (z' + i\gamma')^{k+1+in}\, (z' - i\gamma')^{k+1-in'} \quad (3.21)$$

The integral with respect to z is found by choosing for C the contour shown below, consisting of parts of the axes and parts of a circle whose radius $\to \infty$. It can then be compounded of two parts C_1 and C_2. C_1 consists of two circles at infinity, on which the integrand $\to 0$ like z^{-2}, and two straight parts along the real axis in opposite directions, on which the integrand has the same value. Thus the integral round C_1 vanishes. C_2 does not enclose the branch points $\pm i\gamma$, but consists of three closed curves, of which one encircles the pole $\varepsilon - z'$ positively and one negatively. On the

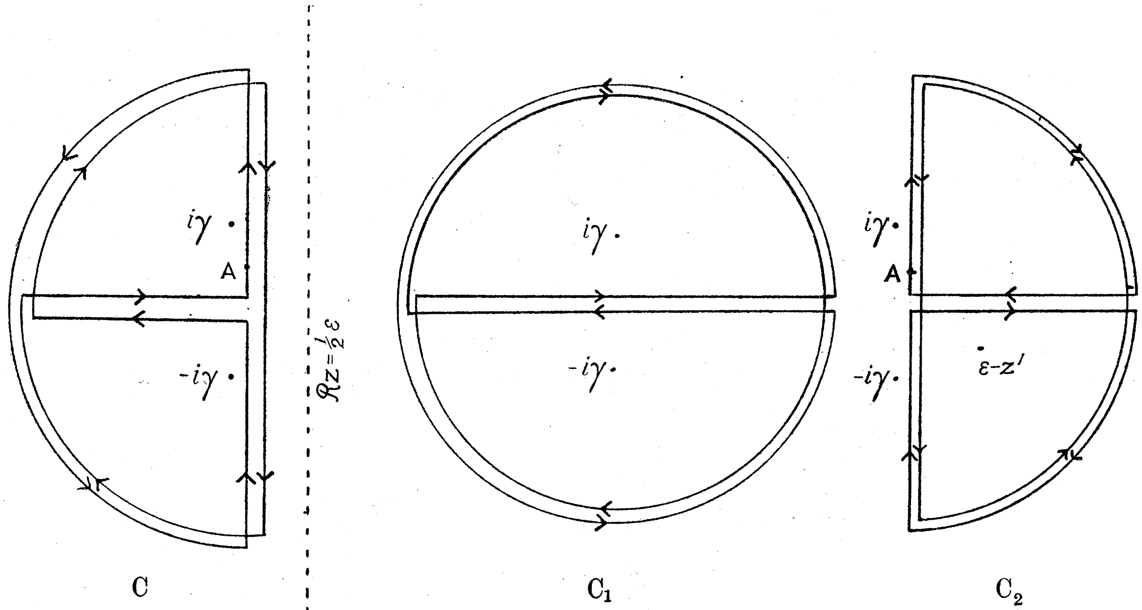

FIG. 4.

latter $|\arg(z \pm i\gamma)| < \pi$, but on the former the integrand is multiplied by $e^{-2\pi n}$. Thus by the theory of residues

$$\int_C \frac{(2k+1)!}{(\varepsilon - z - z')^{2k+2}} (z+i\gamma)^{k+in}(z-i\gamma)^{k-in}\,dz$$

$$= -(1-e^{-2\pi n})\,2\pi i\left[\left(\frac{d}{dz}\right)^{2k+1}(z+i\gamma)^{k+in}(z-i\gamma)^{k-in}\right]_{z=\varepsilon-z'}, \quad . \quad . \quad (3.22)$$

where

$$|\arg(z \pm i\gamma)| < \pi.$$

Also,

$$\left(\frac{d}{dz}\right)^{2k+1}(z+i\gamma)^{k+in}(z-i\gamma)^{k-in}$$

$$= \sum_{r=0}^{2k+1}\binom{2k+1}{r}\frac{\Gamma(k+1+in)}{\Gamma(k+1-r+in)}(z+i\gamma)^{k-r+in}\frac{\Gamma(k+1-in)}{\Gamma(-k+r-in)}(z-i\gamma)^{-k-1+r-in}$$

$$= \frac{(-)^{k+1}\,i\sinh\pi n}{\pi}|\,\Gamma(k+1-in)\,|^2\,\frac{(z+i\gamma)^{k+in}}{(z-i\gamma)^{k+1+in}}\sum_{r=0}^{2k+1}\binom{2k+1}{r}(-)^r\left(\frac{z-i\gamma}{z+i\gamma}\right)^r$$

$$= \frac{e^{\pi n}(1-e^{-2\pi n})\,|\,\Gamma(k+1-in)\,|^2\,(2\gamma)^{2k+1}}{2\pi\,(z-i\gamma)^{k+1+in}(z-i\gamma)^{k+1-in}}.$$

Thus (3.22) is

$$\frac{-ie^{\pi n}(1-e^{-2\pi n})^2\,|\,\Gamma(k+1-in)\,|^2\,(2\gamma)^{2k+1}}{(z'-\varepsilon+i\gamma)^{k+1+in}(z'-\varepsilon-i\gamma)^{k+1-in}} \quad \left(0 < \arg(z'-\varepsilon \mp i\gamma) < 2\pi\right).$$

On substituting this in (3.21) we see that the restriction $\Re z' < \frac{1}{2}\varepsilon$ can be removed if

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C′ passes to the left of $\varepsilon \pm i\gamma$. Provided C′ passes to the left of $\pm i\gamma$, there is now no difficulty in $\varepsilon \to 0$, and (3.20) becomes

$$ie^{\pi n}(1-e^{-2\pi n})^2\,|\,\Gamma(k+1-in)\,|^2\,(2\gamma)^{2k+1}$$
$$\times \int_{C'} \frac{(z'+i\gamma')^{k+1+in'}(z'-i\gamma')^{k+1-in'}}{(z'+i\gamma)^{k+1+in}(z'-i\gamma)^{k+1-in}}\,dz', \quad . \quad . \quad . \quad . \quad (3.23)$$

where

$$0 < \arg(z' \pm i\gamma) < 2\pi, \quad |\arg(z' \pm i\gamma')| < \pi \quad \text{(at A)}.$$

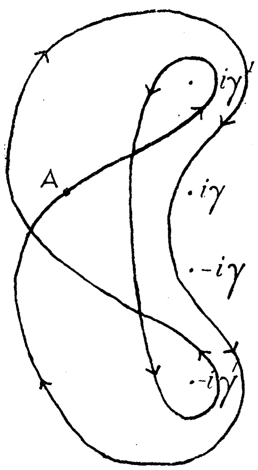

FIG. 5.

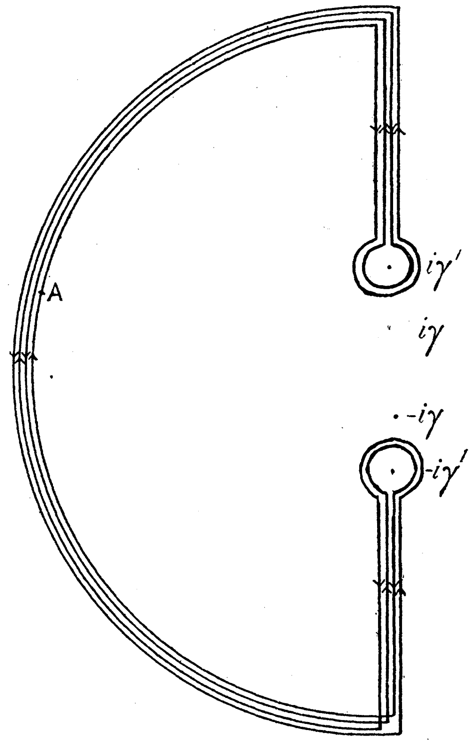

FIG. 6.

The contour C′ can now be deformed into a curve lying on a semicircle of radius R and the imaginary axis, with loops round $\pm i\gamma$. We make $R \to \infty$. At A

$$\arg(z'-i\gamma') \sim -\pi\,; \quad \arg(z'+i\gamma') \sim \pi\,;$$

and the integrand $\dot{\sim}\ e^{-2\pi n'}$. On the imaginary axis we put $z' = \pm iy$. We find

$$\int_{C'} \frac{(z'+i\gamma')^{k+1+in'}(z'-i\gamma')^{k+1-in'}}{(z'+i\gamma)^{k+1+in}(z'-i\gamma)^{k+1-in}}\,dz'$$
$$= (-2e^{-2\pi n'}+1+e^{-4\pi n'})\int_{\gamma'}^{R} \frac{(y+\gamma')^{k+1+in'}(y-\gamma')^{k+1-in'}}{(y+\gamma)^{k+1+in}(y-\gamma)^{k+1-in}}\,i\,dy$$
$$+ (2e^{-2\pi n'}-1-e^{-4\pi n'})\int_{\gamma'}^{R} \frac{(y-\gamma')^{k+1+in'}(y+\gamma')^{k+1-in'}}{(y-\gamma)^{k+1+in}(y+\gamma)^{k+1-in}}\,(-i)\,dy$$
$$+ (2e^{-2\pi n'}-1-e^{-4\pi n'})\,2iR + O(1/R),$$
$$= 2i\gamma\,(1-e^{-2\pi n'})^2\,\mathfrak{R}\mathrm{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.24)$$

where

$$\mathrm{I} = \lim_{\mathrm{R}\to\infty} \frac{1}{\gamma}\left\{\int_{\gamma'}^{\mathrm{R}} \frac{(y+\gamma')^{k+1+in'}(y-\gamma')^{k+1-in'}}{(y+\gamma)^{k+1+in}(y-\gamma)^{k+1-in}}\,dy - \mathrm{R}\right\}. \quad . \quad . \quad . \qquad (3.25)$$

Let

$$\gamma' = \lambda\gamma, \quad \text{so that} \quad \lambda > 1, \quad n = \lambda n'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.26)$$

Put $y = \gamma u$, and integrate by parts, getting

$$\mathrm{I} = \lim_{\mathrm{R}\to\infty} \left\{\int_{\lambda}^{\mathrm{R}/\gamma} \frac{(u+\lambda)^{k+1+in'}(u-\lambda)^{k+1-in'}}{(u+1)^{k+1+i\lambda n'}(u-1)^{k+1-i\lambda n'}}\,du - \mathrm{R}/\gamma\right\},$$

$$= -\int_{\lambda}^{\infty} \frac{(u+\lambda)^{k+in'}(u-\lambda)^{k-in'}}{(u+1)^{k+2+i\lambda n'}(u-1)^{k+2-i\lambda n'}}\, 2(\lambda^2-1)\left[(k+1)u - in\right] u\,du. \quad . \quad . \qquad (3.27)$$

Inserting (3.24) in (3.23), and multiplying by $\xi(\mathrm{E}, k)$, $\xi(\mathrm{E}', k+1)$, from (2.46) we find

$$(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) = -\frac{4\pi m}{h}\frac{e^{\frac{1}{2}\pi n}\,|\,\Gamma(k+1-in)\,|}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k+2-in')\,|}\left(\frac{\gamma}{\gamma'}\right)^{k+\frac{3}{2}} \Re\mathrm{I} \quad (\mathrm{E}' > \mathrm{E} > 0). \quad (3.28)$$

3.3. To find $(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1)$ we make a transformation which is equivalent to expressing the matrix-elements of the acceleration in terms of those of the velocity, with the result

$$\int_0^{\infty} u^*(\mathrm{E}, k)\,\frac{1}{r^2}\,u(\mathrm{E}', k-1)\,r^2\,dr$$

$$= \frac{\gamma'^2-\gamma^2}{\alpha}\lim_{\epsilon\to 0}\int_0^{\infty} u^*(\mathrm{E}, k)\left(\frac{d}{dr} - \frac{k-1}{r}\right)u(\mathrm{E}', k-1)\,.\,e^{-\epsilon r}r^2\,dr. \quad . \quad . \qquad (3.30)$$

This is proved in the same way as the orthogonality formula (2.30). By (2.31)

$$\gamma^2 \lim_{\epsilon\to 0}\int_0^{\infty} u^*(\mathrm{E}, k)\left(\frac{d}{dr} - \frac{k-1}{r}\right)u(\mathrm{E}', k-1)\,.\,e^{-\epsilon r}r^2\,dr$$

$$= -\lim_{\epsilon\to 0}\int_0^{\infty}\left(\frac{d^2}{dr^2} + \frac{\alpha}{r} - \frac{k(k+1)}{r^2}\right)ru^*(\mathrm{E}, k)\,.\left(\frac{d}{dr} - \frac{k}{r}\right)ru(\mathrm{E}', k-1)\,.\,e^{-\epsilon r}\,dr,$$

$$= -\lim_{\epsilon\to 0}\int_0^{\infty} ru^*(\mathrm{E}, k)\left(\frac{d^2}{dr^2} + \frac{\alpha}{r} - \frac{k(k+1)}{r^2}\right)e^{-\epsilon r}\left(\frac{d}{dr} - \frac{k}{r}\right)ru(\mathrm{E}', k-1)\,dr, \quad (3.31)$$

after two integrations by parts. Also

$$\gamma'^2 \lim_{\epsilon\to 0}\int_0^{\infty} u^*(\mathrm{E}, k)\left(\frac{d}{dr} - \frac{k-1}{r}\right)u(\mathrm{E}', k-1)\,.\,e^{-\epsilon r}r^2\,dr$$

$$= -\lim_{\epsilon\to 0}\int_0^{\infty} ru^*(\mathrm{E}, k)\,e^{-\epsilon r}\left(\frac{d}{dr} - \frac{k}{r}\right)\left(\frac{d^2}{dr^2} + \frac{\alpha}{r} - \frac{(k-1)k}{r^2}\right)ru(\mathrm{E}', k-1)\,dr. \qquad (3.32)$$

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The exponential factor in (3.31) may be placed before the differential operator. After subtracting (3.31) from (3.32) we may proceed to the limit under the integral, and obtain (3.30). We now use the recurrence formula (2.15), with $k-1$ for k.

$$\int_0^\infty u^*(\mathrm{E}, k)\frac{1}{r^2}u(\mathrm{E}', k-1)\, r^2\, dr$$

$$= \frac{\gamma'^2-\gamma^2}{2k\alpha}\lim_{\epsilon\to 0}\int_0^\infty u^*(\mathrm{E}, k)\left[-\alpha u(\mathrm{E}', k-1) - u(\mathrm{E}', k)\right]e^{-\epsilon r}r^2\, dr,$$

$$= -\frac{\gamma'^2-\gamma^2}{2k}\lim_{\epsilon\to 0}\int_0^\infty u^*(\mathrm{E}, k)\, u(\mathrm{E}', k-1)\, e^{-\epsilon r}r^2\, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.33)$$

by (2.30). We evaluate (3.33) in the same way as (3.20),

$$\lim_{\epsilon\to 0}\int_0^\infty u^*(\mathrm{E}, k)\, u(\mathrm{E}', k-1)\, e^{-\epsilon r}r^2\, dr$$

$$= -\lim_{\epsilon\to 0}\int_0^\infty dr\int_{\mathrm{C}} dz\int_{\mathrm{C}'} dz'\, e^{-(\epsilon-z-z')r}r^{2k+1}(z+i\gamma)^{k+in}(z-i\gamma)^{k-in}(z'+i\gamma')^{k-1+in'}(z'-i\gamma')^{k-1-in'},$$

$$= -\lim_{\epsilon\to 0}\int_{\mathrm{C}} dz\int_{\mathrm{C}'} dz'\, \frac{(2k+1)\,!}{(\varepsilon-z-z')^{2k+2}}(z+i\gamma)^{k+in}(z-i\gamma)^{k-in}(z'+i\gamma')^{k-1+in'}(z'-i\gamma')^{k-1-in'},$$

$$[\mathfrak{R}z,\ \mathfrak{R}z' < \tfrac{1}{2}\varepsilon]$$

$$= ie^{\pi n}(1-e^{-2\pi n})^2\,|\,\Gamma(k+1-in)\,|^2\,(2\gamma)^{2k+1}\int_{\mathrm{C}'}\frac{(z'+i\gamma')^{k-1+in'}(z'-i\gamma')^{k-1-in'}}{(z'+i\gamma)^{k+1+in}(z'-i\gamma)^{k+1-in}}\,dz'$$

$$[0 < \arg(z' \pm i\gamma) < \pi\,;\quad \mathrm{C}' \text{ to the left of } \pm i\gamma],$$

$$= -2e^{\pi n}(1-e^{-2\pi n})^2(1-e^{-2\pi n'})^2\,|\,\Gamma(k+1-in)\,|^2\,(2\gamma)^{2k+1}\,\mathfrak{R}\mathrm{J}/\gamma^3, \quad . \quad . \quad . \quad . \qquad (3.34)$$

where

$$\mathrm{J} = \int_\lambda^\infty \frac{(u+\lambda)^{k-1+in'}(u-\lambda)^{k-1-in'}}{(u+1)^{k+1+i\lambda n'}(u-1)^{k+1+i\lambda n'}}\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.35)$$

Inserting (3.34) in (3.33) and multiplying by $\xi(\mathrm{E}, k)$, $\xi(\mathrm{E}', k-1)$, from (2.46) we have

$$(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) = \frac{4\pi m}{h}\,\frac{e^{\frac{1}{2}\pi n}\,|\,\Gamma(k+1-in)\,|}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k-in')\,|}\left(\frac{\gamma}{\gamma'}\right)^{k-\frac{1}{2}}\frac{2(\lambda^2-1)}{k}\,\mathfrak{R}\mathrm{J}\quad (\mathrm{E}' > \mathrm{E} > 0). \qquad (3.36)$$

3.4. Now let $\mathrm{E} < 0 < \mathrm{E}'$. By (2.11), (2.12),

$$\int_0^\infty u^*(\mathrm{E}, k)\frac{1}{r^2}u(\mathrm{E}', k+1)\, r^2\, dr$$

$$= -\int_0^\infty dr\int_{\mathrm{C}'} dz\sum_{q=0}^{n-k-1}\frac{(n+k)\,!^2\,(-2\gamma)^q r^{2k+1+q}}{q\,!\,(2k+1+q)\,!\,(n-k-1-q)\,!}\,e^{-(\gamma-z)r}(z+i\gamma')^{k+1+in'}(z-i\gamma')^{k+1-in'} \qquad (3.40)$$

If we choose C′ so that on it $\mathfrak{R}z < \gamma$, we can invert the order of integration, and obtain for (3.40)

$$-\int_{C'} \sum_{q=0}^{n-k-1} \frac{(n+k)\,!^2\,(-2\gamma)^q}{q\,!\,(n-k-1-q)\,!\,(\gamma-z)^{2k+2+q}} (z+i\gamma')^{k+1+in'} (z-i\gamma')^{k+1-in'}\,dz$$
$$= -\frac{(n+k)\,!^2}{(n-k-1)\,!} \int_{C'} \frac{(z+\gamma)^{n-k-1}}{(z-\gamma)^{n+k+1}} (z+i\gamma')^{k+1+in'} (z-i\gamma')^{k+1-in'}\,dz. \quad . \quad (3.41)$$

We may now take for C′ the contour used in evaluating (3.23). Then (3.41) becomes

$$-2i\gamma \frac{(n+k)\,!^2}{(n-k-1)\,!} (1-e^{-2\pi n'})^2\, \mathfrak{R}\mathrm{I}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.42)$$

where

$$\mathrm{I}' = \lim_{\mathrm{R}\to\infty} \left\{ \int_{\lambda}^{\mathrm{R}/\gamma} \frac{(u-i)^{n-k-1}}{(u+i)^{n+k+1}} (u+\lambda)^{k+1+in'} (u-\lambda)^{k+1-in'}\,du - \mathrm{R}/\gamma \right\}$$
$$= -\int_{\lambda}^{\infty} \frac{(u-i)^{n-k-2}}{(u+i)^{n+k+2}} (u+\lambda)^{k+in'} (u-\lambda)^{k-in'}\, 2\,(\lambda^2+1)\,[(k+1)\,u - in]\,u\,du. \quad (3.43)$$

Here, as before, $\lambda = \gamma'/\gamma$, but it may be less than 1.

Multiplying (3.42) by ξ (E, k), ξ (E′, $k+1$) from (2.41), (2.46) we have

$$(\mathrm{E}, k|\, 1/r^2 |\mathrm{E}', k+1) = -2i\gamma \sqrt{\left(\frac{4\pi m}{h}\right)} \left[\frac{(n+k)\,!}{(n-k-1)\,!\; 2n}\right]^{\frac{1}{2}} \frac{(\gamma/\gamma')^{k+\frac{3}{2}}}{e^{\frac{1}{2}\pi n'}|\Gamma\,(k+2-in')|}\, \mathfrak{R}\mathrm{I}'$$
$$(\mathrm{E} < 0 < \mathrm{E}'). \quad . \quad . \quad (3.44)$$

3.5. For (E, $k|1/r^2|$E′, $k-1$) we use a formula similar to (3.30), which is proved in exactly the same way. The exponential factor is now unnecessary, and γ^2 is replaced by $-\gamma^2$. We have

$$\int_0^{\infty} u^*\,(\mathrm{E}, k)\; 1/r^2\; u\,(\mathrm{E}', k-1)\, r^2 dr$$
$$= \frac{\gamma'^2+\gamma^2}{\alpha} \int_0^{\infty} u^*\,(\mathrm{E}, k) \left(\frac{d}{dr} - \frac{k-1}{r}\right) u\,(\mathrm{E}', k-1)\,.\, r^2\, dr,$$
$$= -\frac{\gamma'^2+\gamma^2}{2k} \int_0^{\infty} u^*\,(\mathrm{E}, k)\; u\,(\mathrm{E}', k-1)\, r^2\, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.50)$$

by (2.15) and the orthogonality relation. Also

$$\int_0^{\infty} u^*\,(\mathrm{E}, k)\; u\,(\mathrm{E}', k-1)\, r^2\, dr$$
$$= -\int_0^{\infty} dr \int_{C'} dz \sum_{q=0}^{n-k-1} \frac{(n+k)\,!^2\,(-2\gamma)^q\, r^{2k+1+q}}{q\,!\,(2k+1+q)\,!\,(n-k-1-q)\,!}$$
$$\times\, e^{-(\gamma-z)r} (z+i\gamma')^{k-1+in'} (z-i\gamma')^{k-1-in'},$$
$$= -\frac{(n+k)\,!^2}{(n-k-1)\,!} \int_{C'} \frac{(z+\gamma)^{n-k-1}}{(z-\gamma)^{n+k+1}} (z+i\gamma')^{k-1+in'} (z-i\gamma')^{k-1-in'}\,dz, \quad . \quad . \quad (3.51)$$

as before, provided C′ passes to the left of γ. Using the same contour as before, we find for (3.51)

$$-\frac{2i}{\gamma^3}\frac{(n+k)!^2}{(n-k-1)!}(1-e^{-2\pi n'})^2\,\Re J', \quad . \quad . \quad . \quad . \qquad (3.52)$$

where

$$J' = \int_\lambda^\infty \frac{(u-i)^{n-k-1}}{(u+i)^{n+k+1}}(u+\lambda)^{k-1+in'}(u-\lambda)^{k-1-in'}\,du. \qquad (3.53)$$

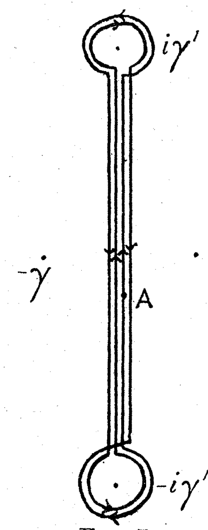

Fig. 7.

Substituting (3.52) in (3.50), and multiplying by ξ (E, k), ξ (E′, k—1), we have

$$(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) = 2i\gamma\sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)!}{(n-k-1)!\,2n}\right]^{\frac{1}{2}}$$

$$\times\frac{(\gamma/\gamma')^{k-\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k-in')\,|}\,\frac{2(\lambda^2+1)}{k}\,\Re J' \quad (\mathrm{E}<0<\mathrm{E}'). \qquad (3.54)$$

3.6. An alternative way of evaluating (3.41), (3.51) is to make C′ lie on the imaginary axis between $\pm\, i\gamma'$, with loops round these points. Then, putting $z = i\gamma$, $y = \gamma' v$, we have

$$\int_{C'}\frac{(z+\gamma)^{n-k-1}}{(z-\gamma)^{n+k+1}}(z+i\gamma')^{k+1+in'}(z-i\gamma')^{k+1-in'}\,dz$$

$$= (2e^{-\pi n'} - e^{\pi n'} - e^{-3\pi n'})\,(-)^{k+1}\int_{-\gamma'}^{\gamma'}\frac{(y-i\gamma)^{n-k-1}}{(y+i\gamma)^{n+k+1}}(y+\gamma')^{k+1+in'}(\gamma'-y)^{k+1-in'}\,i\,dy,$$

$$= (-)^{n+k}\,ie^{\pi n'}(1-e^{-2\pi n'})^2\,\frac{\gamma'^{2k+3}}{\gamma^{2k+2}}\,\mathrm{I}'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.60)$$

where

$$\mathrm{I}'' = \int_{-1}^{1}\frac{(1+i\lambda v)^{n-k-1}}{(1-i\lambda v)^{n+k+1}}(1+v)^{k+1+in'}(1-v)^{k+1-in'}\,dv. \quad . \quad . \quad . \qquad (3.61)$$

Similarly

$$\int_{C}\frac{(z+\gamma)^{n-k-1}}{(z-\gamma)^{n+k+1}}(z+i\gamma')^{k-1+in'}(z-i\gamma')^{k-1-in'}\,dz$$

$$= (-)^{n+k}\,ie^{\pi n'}(1-e^{-2\pi n'})^2\,\frac{\gamma'^{2k-1}}{\gamma^{2k+2}}\,\mathrm{J}'', \quad . \quad . \quad . \quad . \qquad (3.62)$$

where

$$\mathrm{J}'' = \int_{-1}^{1}\frac{(1+i\lambda v)^{n-k-1}}{(1-i\lambda v)^{n+k+1}}(1+v)^{k-1+in'}(1-v)^{k-1-in'}\,dv. \quad . \quad . \quad . \quad . \qquad (3.63)$$

Using (3.62), (3.63), we find

$$(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) = (-)^{n+k+1}\,i\gamma\sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)!}{(n-k-1)!\,2n}\right]^{\frac{1}{2}}$$

$$\times\frac{e^{\frac{1}{2}\pi n'}}{|\,\Gamma(k+2-in')\,|}\left(\frac{\gamma'}{\gamma}\right)^{k+\frac{3}{2}}\mathrm{I}'', . \qquad (3.64)$$

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$$(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) = (-)^{n+k}\, i\gamma \sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)\,!}{(n-k-1)\,!\;2n}\right]^{\frac{1}{2}}$$
$$\times \frac{e^{\frac{1}{2}\pi n'}}{|\Gamma(k-in')|}\left(\frac{\gamma'}{\gamma}\right)^{k-\frac{1}{2}} \frac{2(\lambda^2+1)}{k}\,\mathrm{J}'', \quad (3.65)$$
$$(\mathrm{E} < 0 < \mathrm{E}').$$

§ 4. *Approximations to the Matrix-Elements.*

4.0. We first suppose that $|\mathrm{E}|$ and E' are small, and that their ratio is not large, nor (if $\mathrm{E} > 0$) too near 1. Thus we make n and $n' \to \infty$ in a fixed ratio λ. The integrals (3.27), (3.35) may be written

$$\left.\begin{aligned} \mathrm{I} &= -\int_\lambda^\infty \left[\frac{u^2-\lambda^2}{u^2-1}\right]^{k+1}\left[\frac{(u+\lambda)(u-1)^\lambda}{(u-\lambda)(u+1)^\lambda}\right]^{in'} \frac{2(\lambda^2-1)\,\{(k+1)\,u-in\}\,u\,du}{(u^2-\lambda^2)(u^2-1)} \\ \mathrm{J} &= \int_\lambda^\infty \left[\frac{u^2-\lambda^2}{u^2-1}\right]^{k}\left[\frac{(u+\lambda)\,(u-1)^\lambda}{(u-\lambda)\,(u+1)^\lambda}\right]^{in'} \frac{du}{(u^2-\lambda^2)\,(u^2-1)} \end{aligned}\right\}. \quad (4.00)$$

Let*

$$\frac{(u+\lambda)\,(u-1)^\lambda}{(u-\lambda)\,(u+1)^\lambda} = e^{x^3/n}. \quad \ldots \quad (4.01)$$

Then

$$\frac{3x^2\,dx}{n'} = -\frac{2\lambda\,(\lambda^2-1)\,du}{(u^2-\lambda^2)\,(u^2-1)}, \quad \ldots \quad (4.02)$$

and

$$\left.\begin{aligned} \mathrm{I} &= -\int_0^\infty \left[\frac{u^2-\lambda^2}{u^2-1}\right]^{k+1} e^{ix^3}\frac{\{(k+1)\,u-in\}\,u\,.\,3x^2\,dx}{n} \\ \mathrm{J} &= \int_0^\infty \left[\frac{u^2-\lambda^2}{u^2-1}\right]^{k} e^{ix^3}\frac{3x^2\,dx}{2n\,(\lambda^2-1)} \end{aligned}\right\}, \quad \ldots \quad (4.03)$$

We integrate J by parts, using

$$\frac{d}{dx}\left[\frac{u^2-\lambda^2}{u^2-1}\right]^k = \left[\frac{u^2-\lambda^2}{u^2-1}\right]^k \frac{2ku\,(\lambda^2-1)}{(u^2-\lambda^2)\,(u^2-1)}\,\frac{du}{dx} = -\left[\frac{u^2-\lambda^2}{u^2-1}\right]^k \frac{ku}{n}\,3x^2\,dx.$$

Thus

$$\mathrm{J} = \frac{+\,i}{2n\,(\lambda^2-1)} - \frac{ik}{2n^2\,(\lambda^2-1)}\int_0^\infty \left[\frac{u^2-\lambda^2}{u^2-1}\right]^k e^{ix^3}\,3ux^2\,dx. \quad \ldots \quad (4.04)$$

We shall assume that only those values of x for which u is large make important contributions to I and J. If k, as well as n and n', $\to \infty$, this is evidently true. If it is not true when k remains finite, then in these cases I remains finite and $\mathfrak{R}\mathrm{J}$ is $\mathrm{O}\,(1/n^2)$. The results we shall obtain are of higher order than these $[\mathfrak{R}\mathrm{I} = \mathrm{O}\,(n^{\frac{1}{3}}),\ \mathfrak{R}\mathrm{J} = \mathrm{O}\,(n^{-\frac{5}{3}})]$, and even so, the corresponding matrix-elements are negligible. Thus our assumption is correct except in negligible cases.

* I am indebted to Prof. LITTLEWOOD for suggesting a transformation similar to this.

We see from (4.02) that for large u

$$\frac{x^3}{n'} \sim \frac{2\lambda(\lambda^2-1)}{3u^3}, \qquad (4.05)$$

and

$$\left[\frac{u^2-\lambda^2}{u^2-1}\right]^k \sim \left[1-\frac{(\lambda^2-1)x^2}{\{\tfrac{2}{3}n(\lambda^2-1)\}^{2/3}}\right]^k,$$

$$\sim e^{-3ax^2}, \qquad (4.06)$$

where

$$a = \left(\frac{\lambda^2-1}{3}\right)^{1/3}\frac{k}{(2n)^{2/3}}, \qquad (4.07)$$

so that

$$2a^3 = \tfrac{1}{6}k^3\left(\frac{1}{n'^2}-\frac{1}{n^2}\right) = \frac{(\mathrm{E}'-\mathrm{E})\,h^2k^3}{12\pi^2 m Z^2 e^4}. \qquad (4.08)$$

The final approximation (4.06) is a good one for large u unless $3ax^2$ is large. It is then too large, but even so, the contribution of such values of x to the integral is negligible. Since only large values of k are important, the distinction between k and $k+1$ is negligible. Using (4.05), (4.06) in (4.03), (4.04), we find

$$\left.\begin{aligned} \mathrm{I} &\sim -\frac{2na}{k}\int_0^\infty e^{-3ax^2+ix^3}(6a-3ix)\,dx \\ \mathrm{J} &\sim \frac{i}{2n(\lambda^2-1)} - \frac{ia}{n(\lambda^2-1)}\int_0^\infty e^{-3ax^2+ix^3}\,3x\,dx \end{aligned}\right\} \qquad (4.09)$$

4.1. These integrals can be evaluated in terms of Bessel functions. Let

$$\left.\begin{aligned} f(a) &\equiv \int_{-\infty}^{\infty} e^{-3ax^2+ix^3}(ix-a)\,dx \\ g(a) &= \int_{-\infty}^{\infty} e^{-3ax^2+ix^3}\,a\,dx \end{aligned}\right\}, \qquad (4.10)$$

so that

$$\left.\begin{aligned} \Re\mathrm{I} &\sim \frac{3na}{k}[f(a)-g(a)] \\ \Re\mathrm{J} &\sim \frac{-3a}{2n(\lambda^2-1)}[f(a)+g(a)] \end{aligned}\right\} \qquad (4.11)$$

Consider

$$\int_\Gamma e^{-3az^2+iz^3}(pz+q)\,dz,$$

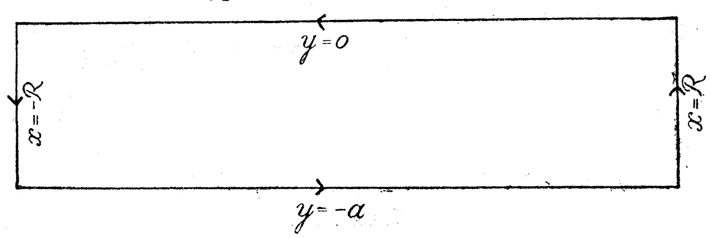

FIG. 8.

where Γ is the rectangle with sides $\Re z = \pm R$, $\Im z = 0, -ia$. On the first side we may put $z = R + iy$. The modulus of the integrand is less than

$$e^{-3.(a+y)R^2+2a^3} 2pR.$$

Integrating from $y = -a$ to $y = 0$, we see that the part of the integral contributed by this side vanishes as $R \to \infty$. Similarly for $\Re z = -R$. Since the integrand is regular in the rectangle, the integrals along the other two sides must cancel in the limit. Thus

$$\int_{-\infty}^{\infty} e^{-3ax^2+ix^3}(px+q)\,dx = \int_{-\infty}^{\infty} e^{ix^3+3ia^2x+2a^3}(px - ipa + q)\,dx. \quad . \quad . \quad (4.12)$$

Hence

$$\left.\begin{aligned} f(a) &= \int_{-\infty}^{\infty} e^{ix^3+3ia^2x+2a^3}\, ix\,dx = -2e^{2a^3}\int_0^{\infty} x\sin(x^3+3a^2x)\,dx \\ g(a) &= \int_{-\infty}^{\infty} e^{ix^3+3ia^2x+2a^3}\, a\,dx = 2e^{2a^3}\int_0^{\infty} a\cos(x^3+3a^2x)\,dx \end{aligned}\right\} . \quad . \quad (4.13)$$

Now in terms of Bessel functions,*

$$\int_0^{\infty}\cos(x^3+3a^2x)\,dx = \frac{\pi}{3}a\,\{I_{1/3}(2a^3) - I_{-1/3}(2a^3)\}, \quad . \quad . \quad . \quad . \quad (4.14)$$

and by differentiation with respect to a,†

$$\int_0^{\infty} x\sin(x^3+3a^2x)\,dx = \frac{\pi}{3}a^2\,\{I_{2/3}(2a^3) - I_{-2/3}(2a^3)\}. \quad . \quad . \quad . \quad . \quad (4.15)$$

Thus

$$\left.\begin{aligned} f(a) &= -\tfrac{2}{3}\pi a^2 e^{2a^3}F(a) \\ g(a) &= \tfrac{2}{3}\pi a^2 e^{2a^3}G(a) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.16)$$

where

$$\left.\begin{aligned} F(a) &= I_{\frac{2}{3}}(2a^3) - I_{-\frac{2}{3}}(2a^3) \\ G(a) &= I_{\frac{1}{3}}(2a^3) - I_{-\frac{1}{3}}(2a^3) \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.17)$$

It is convenient to know the behaviour of F and G as $a \to 0$ or ∞. As $z \to 0$‡

$$I_p(z) = i^{-p}J_p(iz) \sim (\tfrac{1}{2}z)^p/\Gamma(p+1).$$

Thus

$$F(a) \sim -a^{-2}/\Gamma(\tfrac{1}{3})\,; \quad G(a) \sim -a^{-1}/\Gamma(\tfrac{2}{3}) \qquad (a \to 0). \quad . \quad . \quad . \quad (4.18)$$

As $z \to \infty$ ||

$$I_n(z) - I_{-n}(z) \sim \left[e^{-(n+\frac{1}{2})\pi i} - e^{-(-n+\frac{1}{2})\pi i}\right]\frac{e^{-z}}{(2\pi z)^{\frac{1}{2}}}.$$

* NICHOLSON, 'Phil. Mag.,' vol. 18, p. 6 (1909).

† Using $\frac{d}{dz}(z^{\pm n}I_n) = z^{\pm n}I_{n\mp 1}$.

‡ JAHNKE und EMDE, 'Funktionentafeln,' p. 90.

|| WHITTAKER and WATSON, 'Modern Analysis,' p. 373.

Thus

$$\mathrm{F}(a) \sim -e^{-2a^3}\sqrt{\frac{3}{4\pi a^3}}; \quad \mathrm{G}(a) \sim -e^{-2a^3}\sqrt{\frac{3}{4\pi a^3}} \qquad (a \to \infty). \tag{4.19}$$

4.2. Inserting (4.16) in (4.11), we have

$$\left.\begin{aligned} \Re\mathrm{I} &\sim -\frac{2n}{k}\pi a^3 e^{2a^3}(\mathrm{F}+\mathrm{G}) \\ \Re\mathrm{J} &\sim \frac{\pi a^3}{n(\lambda^2-1)} e^{2a^3}(\mathrm{F}-\mathrm{G}) \end{aligned}\right\} \tag{4.20}$$

When k remains finite as n and $n' \to \infty$, $a \to 0$ like $n^{-2/3}$; and (4.18), (4.20) show that $\Re\mathrm{I}$ is $\mathrm{O}(n^{1/3})$ and $\Re\mathrm{J}$i s $\mathrm{O}(n^{-2/3})$, as was mentioned above. Inserting (4.20) in (3.28), (3.36), we have

$$\left.\begin{aligned} (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) &\sim \frac{4\pi m}{h}\,\frac{e^{\frac{1}{2}\pi n}\,|\,\Gamma(k+1-in)\,|\,n'^{k+\frac{3}{2}}}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k+2-in')\,|\,n^{k+\frac{1}{2}}}\,\frac{2\pi a^3}{k}\,e^{2a^3}(\mathrm{F}+\mathrm{G}) \\ (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) &\sim \frac{4\pi m}{h}\,\frac{e^{\frac{1}{2}\pi n}\,|\,\Gamma(k+1-in)\,|\,n'^{k-\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k-in')\,|\,n^{k+\frac{1}{2}}}\,\frac{2\pi a^3}{k}\,e^{2a^3}(\mathrm{F}-\mathrm{G}) \end{aligned}\right\} \tag{4.21}$$

Since n and n' are large, we may approximate to the Γ-functions by means of STIRLING's formula. We find

$$\begin{aligned} \log \frac{e^{\frac{1}{2}\pi n}\,|\,\Gamma(k_1-in)\,|\,n'^{k_2-\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma(k_2-in')\,|\,n^{k_1-\frac{1}{2}}} &\sim \tfrac{1}{2}(k_1-\tfrac{1}{2})\log(1+k_1^2/n^2) + n\tan^{-1}k_1/n - k_1 \\ &\quad - \tfrac{1}{2}(k_2-\tfrac{1}{2})\log(1+k_2^2/n'^2) - n'\tan^{-1}k_2/n' + k_2. \end{aligned} \tag{4.22}$$

By (4.07), a is finite when k is $\mathrm{O}(n^{2/3})$. We can then expand (4.22) in powers of k_1/n, k_2/n'. More generally if $k_1, k_2 \ll n^{4/5}$, (4.22) becomes

$$\tfrac{1}{6}\frac{k_1^3}{n^2} - \tfrac{1}{6}\frac{k_2^3}{n'^2} \sim -\tfrac{1}{6}k^3\left(\frac{1}{n'^2}-\frac{1}{n^2}\right) = -2a^3,$$

on putting in the values of k_1 and k_2 required, and using (4.08). Thus

$$\left.\begin{aligned} (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) &\sim \frac{4\pi m}{h}\,\frac{2\pi a^3}{k}(\mathrm{F}+\mathrm{G}) \\ (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) &\sim \frac{4\pi m}{h}\,\frac{2\pi a^3}{k}(\mathrm{F}-\mathrm{G}) \end{aligned}\right\} \quad \begin{aligned}[\mathrm{E}' > \mathrm{E} > 0,\ n = \lambda n' \to \infty \\ 1 \ll k \ll n^{4/5}]. \end{aligned} \tag{4.23}$$

By (4.19) these are negligible for large a. When k is comparable with n, the last step in the approximation breaks down. When $k \gg n^2$ we may expand (4.22) in powers of n/k_1, n'/k_2, obtaining

$$\begin{aligned} &(k_1-\tfrac{1}{2})\log k_1/n + \tfrac{1}{2}\pi n - k_1 - (k_2-\tfrac{1}{2})\log k_2/n' - \tfrac{1}{2}\pi n' + k_2 \\ &\sim -\log k/n' - (k+\tfrac{1}{2})\log\lambda + \tfrac{1}{2}\pi(n-n') \qquad (k_1 = k+1,\ k_2 = k+2), \\ &\sim \log k/n' - (k+\tfrac{1}{2})\log\lambda + \tfrac{1}{2}\pi(n-n') \qquad (k_1 = k+1,\ k_2 = k), \end{aligned}$$

so that

$$\left.\begin{aligned}(\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) &\sim \frac{4\pi m}{h}\frac{2\pi a^3}{k}\frac{n'}{k}\frac{e^{\frac{1}{2}\pi(n-n')+2a^3}}{\lambda^{k+\frac{1}{2}}}(\mathrm{F}+\mathrm{G})\\ (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) &\sim \frac{4\pi m}{h}\frac{2\pi a^3}{k}\frac{k}{n'}\frac{e^{\frac{1}{2}\pi(n-n')+2a^3}}{\lambda^{k+\frac{1}{2}}}(\mathrm{F}-\mathrm{G})\end{aligned}\right\}\quad [k \gg n^2]. \quad . \quad . \quad . \quad (4.24)$$

These again are very small, though not as small as (4.23). So long as n and n' are sufficiently large, (4.23) is a good approximation for all matrix-elements which are not negligible.

4.3. The matrix-elements with $\mathrm{E} < 0$ can be treated in the same way when $n, n' \to \infty$. From (3.43), (3.53),

$$\left.\begin{aligned}\mathrm{I}' &= -\int_{\lambda}^{\infty}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k+1}\left[\frac{(u+\lambda)(u+i)^{i\lambda}}{(u-\lambda)(u-i)^{i\lambda}}\right]^{in'}\frac{2(\lambda^2+1)\{(k+1)u-in\}\,u\,du}{(u^2-\lambda^2)(u^2+1)}\\ \mathrm{J}' &= \int_{\lambda}^{\infty}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k}\left[\frac{(u+\lambda)(u+i)^{i\lambda}}{(u-\lambda)(u-i)^{i\lambda}}\right]^{in'}\frac{du}{(u^2-\lambda^2)(u^2+1)}\end{aligned}\right\}. \quad (4.30)$$

Let

$$e^{x^3/n'} = \frac{(u+\lambda)(u+i)^{i\lambda}}{(u-\lambda)(u-i)^{i\lambda}} = \frac{u+\lambda}{u-\lambda}e^{-2\lambda\cot^{-1}u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.31)$$

so that

$$\frac{3x^2\,dx}{n'} = -\frac{2\lambda(\lambda^2+1)\,du}{(u^2-\lambda^2)(u^2+1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.32)$$

Then

$$\mathrm{I}' = -\int_0^{\infty}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k+1}e^{ix^3}\{(k+1)u-in\}\,u\frac{3x^2\,dx}{n}, \quad . \quad . \quad . \quad . \quad . \quad (4.33)$$

and

$$\begin{aligned}\mathrm{J}' &= \int_0^{\infty}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k}e^{ix^3}\frac{3x^2\,dx}{2n(\lambda^2+1)},\\ &= \frac{i}{2n(\lambda^2+1)} - \frac{i}{2n^2(\lambda^2+1)}\int_0^{\infty}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k}e^{ix^3}\,3ux^2\,dx \quad . \quad . \quad . \quad (4.34)\end{aligned}$$

after an integration by parts. As before, in all important cases we need consider only large values of u, so that

$$\frac{x^3}{n'} \sim \frac{2\lambda(\lambda^2+1)}{3u^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.35)$$

and

$$\begin{aligned}\left[\frac{u^2-\lambda^2}{u^2+1}\right]^{k} &\sim \left[1-\frac{(\lambda^2+1)x^2}{\{\frac{2}{3}n(\lambda^2+1)\}^{\frac{2}{3}}}\right]^{k},\\ &\sim e^{-3ax^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.36)\end{aligned}$$

where

$$a = \left(\frac{\lambda^2+1}{3}\right)^{\frac{1}{3}}\frac{k}{(2n)^{\frac{2}{3}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.37)$$

or

$$2a^3 = \tfrac{1}{6}k^3\left(\frac{1}{n'^2}+\frac{1}{n^2}\right) = \frac{(\mathrm{E}'-\mathrm{E})\,h^2k^3}{12\pi^2 mZ^2e^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.38)$$

Using (4.35), (4.36) in (4.33), (4.34), we have

$$\left.\begin{aligned} \mathrm{I}' &\sim \frac{2na}{k}\int_0^\infty e^{-3ax^2+ix^3}\,(6a-3ix)\,dx \\ \mathrm{J}' &\sim \frac{i}{2n\,(\lambda^2+1)} - \frac{ia}{n\,(\lambda^2+1)}\int_0^\infty e^{-3ax^2+ix^3}\,3x\,dx \end{aligned}\right\}, \qquad (4.39)$$

These equations are the same as (4.09), with λ^2+1 for λ^2-1. They lead to equations parallel to (4.20)

$$\left.\begin{aligned} \Re\mathrm{I} &\sim -\frac{2n}{k}\,\pi a^3 e^{2a^3}\,(\mathrm{F}+\mathrm{G}) \\ \Re\mathrm{J} &\sim \frac{\pi a^3}{n\,(\lambda^2+1)}\,e^{2a^3}\,(\mathrm{F}-\mathrm{G}) \end{aligned}\right\}. \qquad (4.40)$$

Thus by (3.44), (3.54),

$$\left.\begin{aligned} (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) &\sim \frac{i\alpha}{n}\sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)\,!}{(n-k-1)\,!\,2n}\right]^{\frac{1}{2}} \\ &\qquad\times\frac{n'^{k+\frac{3}{2}}/n^{k+\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}|\,\Gamma\,(k+2-in')\,|}\,\frac{2\pi a^3}{k}\,e^{2a^3}\,(\mathrm{F}+\mathrm{G}) \\ (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) &\sim \frac{i\alpha}{n}\sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)\,!}{(n-k-1)\,!\,2n}\right]^{\frac{1}{2}} \\ &\qquad\times\frac{n'^{k-\frac{1}{2}}/n^{k+\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}|\,\Gamma\,(k-in')\,|}\,\frac{2\pi a^3}{k}\,e^{2a^3}\,(\mathrm{F}-\mathrm{G}) \end{aligned}\right\}. \qquad (4.41)$$

By STIRLING's formula, when $n-k$ is large

$$\begin{aligned} &\log\left\{\left[\frac{(n+k)\,!}{(n-k-1)\,!}\right]^{\frac{1}{2}}\frac{n'^{k'-\frac{1}{2}}/n^{k+\frac{1}{2}}}{e^{\frac{1}{2}\pi n'}\,|\,\Gamma\,(k'-in')\,|}\right\} \\ &\qquad\sim \tfrac{1}{2}\,(k+\tfrac{1}{2})\log\frac{(n^2-k^2)}{n^2} + \tfrac{1}{2}n\log\frac{n+k}{n-k} - k \\ &\qquad - \tfrac{1}{2}\,(k'-\tfrac{1}{2})\log\frac{n'^2+k'^2}{n'^2} - n'\tan^{-1}k'/n' + k' - \tfrac{1}{2}\log 2\pi, \end{aligned} \qquad (4.42)$$

and if $k \ll n^{4/5}$, as when a is finite, (4.42) reduces to

$$-\tfrac{1}{2}\log 2\pi - \tfrac{1}{6}\frac{k^3}{n^2} - \tfrac{1}{6}\frac{k'^3}{n'^2} \sim -2a^3 - \tfrac{1}{2}\log 2\pi$$

by (4.38). Thus

$$\left.\begin{aligned} (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k+1) &\sim i\alpha\sqrt{\frac{m}{hn^3}}\,\frac{2\pi a^3}{k}\,(\mathrm{F}+\mathrm{G}) \\ (\mathrm{E}, k\,|\,1/r^2\,|\,\mathrm{E}', k-1) &\sim i\alpha\sqrt{\frac{m}{hn^3}}\,\frac{2\pi a^3}{k}\,(\mathrm{F}-\mathrm{G}) \end{aligned}\right\} \quad \begin{gathered}[\mathrm{E} < 0 < \mathrm{E}',\ n = \lambda n' \to \infty \\ k \ll n^{\frac{4}{5}}].\end{gathered} \qquad (4.43)$$

The final approximations break down when k approaches n, but the matrix-elements are then negligible.

4.5. We now suppose that $|E|$ only is small. That is, we make n, $\lambda \to \infty$ together, while n' and k remain finite. In (3.27) and (3.35) put $u = \lambda/t$. Then

$$\left.\begin{aligned} I &= -\frac{2(\lambda^2-1)}{\lambda}\int_0^1 \frac{(1+t)^{k+in'}(1-t)^{k-in'}}{(1+t/\lambda)^{k+2+i\lambda n'}(1-t/\lambda)^{k+2-i\lambda n'}}[k+1-in't]\,dt \\ J &= 1/\lambda^3\int_0^1 \frac{(1+t)^{k-1+in'}(1-t)^{k-1-in'}}{(1+t/\lambda)^{k+1+i\lambda n'}(1-t/\lambda)^{k+1-i\lambda n'}}t^2\,dt \end{aligned}\right\} . \quad (4.50)$$

When $\lambda \to \infty$

$$\left.\begin{aligned} I &\sim -2\lambda\int_0^1 (1+t)^{k+in'}(1-t)^{k-in'}e^{-2in't}[k+1-in't]\,dt \\ J &\sim 1/\lambda^3\int_0^1 (1+t)^{k-1+in'}(1-t)^{k-1-in'}e^{-2in't}t^2\,dt \end{aligned}\right\} . \quad (4.51)$$

The same integrals taken from -1 to 1 give $2\mathfrak{R}I$ and $2\mathfrak{R}J$. In these integrals put $t = 2x - 1$ and expand the exponentials in powers of x. Then

$$\begin{aligned} \mathfrak{R}I &\sim -\lambda 2^{2k+1}\int_0^1 x^{k+in'}(1-x)^{k-in'}e^{-4in'x+2in'}[k+1+in'-2in'x]\,dx, \\ &= -\lambda 2^{2k+1}e^{2in'}\Big\{(k+1+in')\sum_{q=0}^{\infty}\frac{\Gamma(k+1+q+in')\,\Gamma(k+1-in')}{(2k+1+q)!}\frac{(-4in')^q}{q!} \\ &\qquad -2in'\sum_{q=0}^{\infty}\frac{\Gamma(k+2+q+in')\,\Gamma(k+1-in')}{(2k+2+q)!}\frac{(-4in')^q}{q!}\Big\}, \\ &= -\lambda 2^{2k+1}e^{2in'}\sum_{q=0}^{\infty}\frac{\Gamma(k+1+q+in')\,\Gamma(k+1-in')}{(2k+2+q)!}\frac{(-4in')^q}{q!}(k+1-in') \\ &\qquad \times(2k+2+q+2in'), \\ &= -\lambda 2^{2k+1}\frac{|\Gamma(k+2-in')|^2}{(2k+2)!}e^{2in'}\{{}_1F_1(k+2+in'\,;\,2k+3\,;\,-4in') \\ &\qquad +{}_1F_1(k+1+in'\,;\,2k+3\,;\,-4in')\}, \\ &= -2\lambda 2^{2k+1}\frac{|\Gamma(k+2-in')|^2}{(2k+2)!}\,\mathfrak{R}\,e^{2in'}{}_1F_1(k+1+in'\,;\,2k+3\,;\,-4in'), \quad (4.52) \end{aligned}$$

since*

$$e^{2in'}{}_1F_1(k+2+in'\,;\,2k+3\,;\,-4in') = e^{-2in'}{}_1F_1(k+1-in'\,;\,2k+3\,;\,4in').$$

Similarly,

$$\begin{aligned} \mathfrak{R}J &\sim \frac{1}{2\lambda^3}2^{2k-1}\int_0^1 x^{k-1+in'}(1-x)^{k-1+in'}e^{-4in'x+2in'}[1-4x(1-x)]\,dx, \\ &= \frac{1}{2\lambda^3}2^{2k-1}e^{2in'}\Big\{\sum_{q=0}^{\infty}\frac{\Gamma(k+q+in')\,\Gamma(k-in')}{(2k-1+q)!}\frac{(-4in')^q}{q!} \\ &\qquad -4\sum_{q=0}^{\infty}\frac{\Gamma(k+1+q+in')\,\Gamma(k+1-in')}{(2k+1+q)!}\frac{(-4in')^q}{q!}\Big\}, \\ &= \frac{1}{2\lambda^3}2^{2k-1}\frac{|\Gamma(k-in')|^2}{(2k+1)!}e^{2in'}\{2k(2k+1)\,{}_1F_1(k+in'\,;\,2k\,;\,-4in') \\ &\qquad -4(k^2+n'^2)\,{}_1F_1(k+1+in'\,;\,2k+2\,;\,-4in')\}. \quad (4.53) \end{aligned}$$

* See BARNES, 'Trans. Camb. Phil. Soc.,' vol. 20, p. 253 (1906).

Inserting (4.52), (4.53) in (3.28), (3.36), and using

$$\Gamma(k+1-in) \sim \sqrt{(2\pi)}\, n^{k+\frac{1}{2}}\, e^{-\frac{1}{2}\pi n}$$

we find

$$\left.\begin{aligned}(+0, k\,|1/r^2|\,\mathrm{E}', k+1) &= \frac{4\pi m}{h}\frac{\sqrt{(2\pi)}\, n'^{k+\frac{1}{2}}\,|\Gamma(k+2-in')|\,2^{2k+2}}{e^{\frac{1}{2}\pi n'}(2k+2)\,!}\,\mathrm{P}(n', k)\\ (+0, k\,|1/r^2|\,\mathrm{E}', k-1) &= \frac{4\pi m}{h}\frac{\sqrt{(2\pi)}\, n'^{k+\frac{1}{2}}\,|\Gamma(k-in')|\,2^{2k-1}}{e^{\frac{1}{2}\pi n'}(2k+1)\,!\;k}\,\mathrm{Q}(n', k)\end{aligned}\right\}, \quad (4.54)$$

where

$$\left.\begin{aligned}\mathrm{P}(n', k) &= \mathfrak{R}\, e^{2in'}\,{}_1\mathrm{F}_1(k+1+in';\; 2k+3;\; -4in')\\ \mathrm{Q}(n', k) &= e^{2in'}\{2k(2k+1)\,{}_1\mathrm{F}_1(k+in';\; 2k;\; -4in')\\ &\quad -4(k^2+n'^2)\,{}_1\mathrm{F}_1(k+1+in';\; 2k+2;\; -4in')\}\end{aligned}\right\}. \quad (4.55)$$

When E is small and negative we put $u = \lambda/t$ in (3.43) and (3.53). Then

$$\left.\begin{aligned}\mathrm{I}' &= -\frac{2(\lambda^2+1)}{\lambda}\int_0^1 \frac{(1-it/\lambda)^{\lambda n'-k-2}}{(1+it/\lambda)^{\lambda n'+k+2}}(1+t)^{k+in'}(1-t)^{k-in'}[k+1-in't]\,dt\\ \mathrm{J}' &= 1/\lambda^3\int_0^1 \frac{(1-it/\lambda)^{\lambda n'-k-1}}{(1+it/\lambda)^{\lambda n'+k+1}}(1+t)^{k-1+in'}(1-t)^{k-1-in'}t^2\,dt\end{aligned}\right\}. \quad (4.56)$$

As $\lambda \to 0$, I′ and J′ reduce to the expressions (4.51) for I and J. In (3.44) and (3.54) we use

$$(n+k)\,!/(n-k-1)\,! \sim n^{2k+1}$$

and find

$$\left.\begin{aligned}(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k+1) &\sim i\frac{\alpha}{4\pi}\sqrt{\frac{h}{mn^3}}\,(+0, k\,|1/r^2|\,\mathrm{E}', k+1)\\ (\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k-1) &\sim i\frac{\alpha}{4\pi}\sqrt{\frac{h}{mn^3}}\,(+0, k\,|1/r^2|\,\mathrm{E}', k-1)\end{aligned}\right\}(\mathrm{E} \to -0). \quad (4.57)$$

4.6. Now let E′ be small, and E finite and negative. We use (3.64), (3.65), and approximate to I″ and J″ as $\lambda \to 0$ and $n' \to \infty$. Let

$$\frac{(1+i\lambda v)^{n-k-1}}{(1-i\lambda v)^{n+k+1}} = \sum_{p=0}^{\infty} a_p\,(i\lambda v)^p, \quad \ldots\ldots\ldots\ldots \quad (4.60)$$

so that

$$a_p = \sum_{q=0}^{n-k-1 \text{ or } p}\binom{n-k-1}{q}\binom{n+k+p-q}{p-q}. \quad \ldots\ldots \quad (4.61)$$

Using the expansion, and putting $v = 2x-1$, we have

$$\begin{aligned}\mathrm{I}'' &= \sum_{p=0}^{\infty} a_p\,(i\lambda)^p\,2^{2k+3}\int_0^1 (2x-1)^p\,x^{k+1+in'}(1-x)^{k+1-in'}\,dx,\\ &= \sum_{p=0}^{\infty} a_p\,(i\lambda)^p\,2^{2k+3}\sum_{q=0}^{\infty}\binom{p}{q}\frac{\Gamma(k+2+q+in')\,\Gamma(k+2+p-q-in')}{(2k+3+p)\,!}(-)^{p-q},\\ &\sim 2^{2k+3}\,\Gamma(k+2+in')\,\Gamma(k+2-in')\,\mathrm{P}'(n, k), \quad \ldots\ldots\ldots\ldots \quad (4.62)\end{aligned}$$

where

$$P'(n, k) = \sum_{p=0}^{\infty} \frac{a_p}{(2k+3+p)!} \sum_{q=0}^{p} \binom{p}{q} (-n)^p,$$

$$= \sum_{p=0}^{\infty} \frac{a_p (-2n)^p}{(2k+3+p)!} \quad \ldots \ldots \ldots \quad (4.63)$$

Similarly

$$J'' \sim 2^{2k-1} \Gamma(k+in') \Gamma(k-in') Q'(n, k), \quad \ldots \ldots \quad (4.64)$$

where

$$Q'(n, k) = \sum_{p=0}^{\infty} \frac{a_p (-2n)^p}{(2k-1+p)!} \quad \ldots \ldots \ldots \quad (4.65)$$

Insert (4.61) in (4.63) and put $p = q + r$. Then

$$P'(n, k) = \sum_{q=0}^{n-k-1} \binom{n-k-1}{q} \frac{(-2n)^q}{(2k+3+q)!} {}_1F_1(n+k+1;\ 2k+4+q;\ -2n),$$

$$= e^{-2n} \sum_{q=0}^{n-k-1} \binom{n-k-1}{q} \frac{(-2n)^q}{(2k+3+q)!} {}_1F_1(-n+k+3+q;\ 2k+4+q;\ 2n),$$

$$= e^{-2n} \sum_{q=0}^{n-k-1} \sum_{p=q}^{\infty} \binom{n-k-1}{q} \frac{(-n+k+3+q)_{p-q}}{(p-q)!} \frac{(-)^q (2n)^p}{(2k+3+p)!} \quad \ldots \quad (4.66)$$

Terms of (4.66) vanish for which $-n+k+3+q \leq 0 \leq -n+k+2+p$, so that if $p \geq n-k-2$ the coefficient of $e^{-2n}(2n)^p$ is

$$\frac{(-)^{n+k}(n-k-1) + (-)^{n-k-1}(p-n+k+2)}{(2k+3+p)!} = \frac{(-)^{n-k}(2n-2k-3-p)}{(2k+3+p)!}.$$

Thus

$$P'(n, k) = (-)^{n-k} e^{-2n} \sum_{p=n-k-2}^{\infty} \left\{ \frac{(2n)^{p+1}}{(2k+3+p)!} - \frac{(2n)^p}{(2k+2+p)!} \right\}$$

$$+ e^{-2n} \sum_{p=0}^{n-k-3} \sum_{q=0}^{p} \binom{n-k-1}{q} \binom{n-k-3-q}{p-q} \frac{(-2n)^p}{(2k+3+p)!},$$

$$= (-)^{n-k-1} e^{-2n} \frac{(2n)^{n-k-2}}{(n+k)!} + e^{-2n} \sum_{p=0}^{n-k-3} \frac{b_p (-2n)^p}{(2k+3+p)!}, \quad \ldots \ldots \quad (4.67)$$

where*

$$b_p = \binom{n-k-1}{p} {}_2F_1(-p,\ n-k-2-p;\ n-k-p;\ -1),$$

$$= \binom{n-k-1}{p} 2^p {}_2F_1(-p,\ 2;\ n-k-p;\ \tfrac{1}{2}),$$

$$= \sum_{q=0}^{p} \binom{n-k-1}{q} (p-q+1)(-)^{p-q} 2^q. \quad \ldots \ldots \ldots \quad (4.68)$$

* For the transformation see BARNES, 'Proc. Lond. Math. Soc.,' vol. 6, p. 150 (1908).

Similarly

$$Q'(n, k) = \sum_{q=0}^{n-k-1} \binom{n-k-1}{q} \frac{(-2n)^q}{(2k-1+q)!} {}_1F_1(n+k+1\,;\; 2k+q\,;\; -2n),$$

$$= e^{-2n} \sum_{q=0}^{n-k-1} \binom{n-k-1}{q} \frac{(-2n)^q}{(2k-1+q)!} {}_1F_1(-n+k-1+q\,; 2k+q\,; 2n),$$

$$= e^{-2n} \sum_{q=0}^{n-k-1} \sum_{p=q}^{n-k+1} \binom{n-k-1}{q} \binom{n-k+1-q}{p-q} \frac{(-2n)^p}{(2k-1+p)!},$$

$$= e^{-2n} \sum_{p=0}^{n-k+1} \frac{C_p(-2n)^p}{(2k-1+p)!}, \qquad \text{(4.69)}$$

where C_p is the coefficient of x^p in $[x+(1+x)]^{n-k-1}(1+x)^2$, so that

$$C_p = \binom{n-k}{p} 2^p + \binom{n-k-1}{p-2} 2^{p-2}. \qquad \text{(4.6T)}$$

From (3.64), (3.65), using (4.62), (4.64) and STIRLING's formula,

$$\left.\begin{aligned} &(E, k\,|1/r^2|+0, k+1) \\ &\quad = (-)^{n-k-1}\, i \sqrt{\left[\left(\frac{4\pi m}{h}\right)\frac{(n+k!)}{(n-k-1)!}\right]^{\frac{1}{2}}}\, 4\sqrt{\pi\alpha}\,(4n)^k\, P'(n,k) \\ &(E, k\,|\,1/r^2\,|+0, k-1) \\ &\quad = (-)^{n-k}\, i \sqrt{\left(\frac{4\pi m}{h}\right)}\left[\frac{(n+k)!}{(n-k-1)!}\right]^{\frac{1}{2}} \frac{8\sqrt{\pi\alpha}}{k}\,(4n)^{k-2}\, Q'(n,k) \end{aligned}\right\}(E<0) \qquad \text{(4,6E)}$$

4.7. It is worthy of note that (4.54) reduces to (4.23) and (4.57) to (4.43) as $E' \to 0$, and (4.6E) reduces to (4.57) as $E \to -0$. It is not immediately obvious that this should be so, as one set of formulæ is suitable when E and E′ have a finite ratio, and the other when their ratio is large. However, if in (4.51) we put

$$\frac{1+t}{1-t} e^{-2t} = e^{x^3/n'}, \qquad \text{(4.70)}$$

we find, after integrating J by parts, that

$$\left.\begin{aligned} I &\sim -2\lambda \int_0^\infty (1-t^2)^{k+1} e^{ix^3}[k+1-in't]\frac{3x^2\,dx}{2n't^2} \\ J &\sim \frac{i}{2n\,\lambda^2} - \frac{ik}{2n^2\,\lambda}\int_0^\infty (1-t^2)^k e^{ix^3}\frac{3x^2\,dx}{t} \end{aligned}\right\}, \qquad \text{(4.71)}$$

Only small values of t are important in (4.71), just as only large values of u are important in (4.04), and approximations analogous to those previously made produce the formulæ (4.09), with

$$a = \frac{k}{3^{\frac{1}{3}}(2n')^{\frac{2}{3}}}, \qquad 2a^3 = k^3/6n'^2. \qquad \text{(4.72)}$$

(4.72) is equivalent to (4.07), (4.08) when $n/n' \to \infty$.

To deal with (4.6E), we turn the definitions (4.63), (4.65) of P′ and Q′ into contour integrals

$$\left.\begin{aligned} -(2n)^{2k+3}\,\mathrm{P}'(n,k) &= \frac{1}{2\pi i}\int_{\mathrm{C}} \frac{(1+1/z)^{n-k-1}}{(1-1/z)^{n+k+1}}\,e^{-2nz}\,\frac{dz}{z^{2k+4}} \\ -(2n)^{2k-1}\,\mathrm{Q}'(n,k) &= \frac{1}{2\pi i}\int_{\mathrm{C}} \frac{(1+1/z)^{n-k-1}}{(1-1/z)^{n+k+1}}\,e^{-2nz}\,\frac{dz}{z^{2k}} \end{aligned}\right\}, \quad \ldots \quad (4.73)$$

where C is a contour on which $|z| > 1$.

After partial integrations equations (4.73) become

$$\left.\begin{aligned} -(2n)^{2k+3}\,\mathrm{P}'(n,k) &= \frac{(-)^{n+k+1}}{2\pi i}\int_{\mathrm{C}} \frac{(1+z)^{n-k-2}}{(1-z)^{n+k+2}}\,e^{-2nz}\,[k+1+nz]\,2\,dz \\ -(2n)^{2k-1}\,\mathrm{Q}'(n,k) &= \frac{(-)^{n+k}}{2\pi i}\int_{\mathrm{C}} \frac{(1+z)^{n-k-1}}{(1-z)^{n+k+1}}\,e^{-2nz}\,\frac{k}{n}\,z\,dz. \end{aligned}\right\}. \quad (4.74)$$

The integrands are regular except at $z = 1$, and the contour may be taken to be the imaginary axis and an infinite semicircle over which the integrals vanish. On the imaginary axis we put

$$\frac{1+z}{1-z}\,e^{-2z} = e^{ix^3/n}. \quad \ldots \quad (4.75)$$

When n is large only small values of $|z|$ are relevant, and the usual approximation gives

$$\left.\begin{aligned} (2n)^{2k+3}\,\mathrm{P}'(n,k) &\sim (-)^{n+k+1}\,\frac{na}{\pi k}\int_{-\infty}^{\infty} e^{-3ax^2+ix^3}\,(6a-3ix)\,dx \\ (2n)^{2k-1}\,\mathrm{Q}'(n,k) &\sim (-)^{n+k+1}\,\frac{ia}{2\pi n}\int_{-\infty}^{\infty} e^{-3ax^2+ix^3}\,3x\,dx \end{aligned}\right\}, \quad (4.76)$$

where

$$a = \frac{k}{3^{\frac{1}{3}}(2n)^{\frac{2}{3}}}, \qquad 2a^3 = k^3/6n^2. \quad \ldots \quad (4.77)$$

This definition of a agrees with 4.37 when $n'/n \to \infty$. The integrals in (4.76) are those of (4.09), and

$$\left.\begin{aligned} \mathrm{P}'(n,k) &\sim (-)^{n+k+1}\,\frac{a^3\,e^{2a^3}}{k\,(2n)^{2k+2}}\,(\mathrm{F}+\mathrm{G}) \\ \mathrm{Q}'(n,k) &\sim (-)^{n+k}\,\frac{2a^3\,e^{2a^3}}{(2n)^{2k}}\,(\mathrm{F}-\mathrm{G}) \end{aligned}\right\}, \quad \ldots \quad (4.78)$$

Using (4.78) and STIRLING's formula in (4.6E) we find (4.43) again.

4.8. So far we have made E or E′ or both $\to 0$. Now let $\mathrm{E}' \to \infty$. If in (4.50) we make $n' \to 0$. $\lambda \to \infty$, we have

$$\left.\begin{aligned} \mathrm{I} &\sim -\,2\lambda\int_0^1 (1-t^2)^k\,(k+1)\,dt = -\,\frac{(2k+2)\,!!}{(2k+1)\,!!}\,\lambda \\ \mathrm{J} &\sim 1/\lambda^3\int_0^1 (1-t^2)^{k-1}\,t^2\,dt = \frac{(2k-2)\,!!}{(2k+1)\,!!}\,\frac{1}{\lambda^3} \end{aligned}\right\}, \quad \ldots \quad (4.80)$$

Thus by (3.28), (3.36),

$$\left.\begin{aligned}(\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}', k+1) &\sim \frac{4\pi m}{h} e^{\frac{1}{2}\pi n} \mid \Gamma(k+1-in) \mid \left(\frac{n'}{n}\right)^{k+\frac{1}{2}} \frac{2^{k+1}}{(2k+1)!!} \\ (\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}', k-1) &\sim \frac{4\pi m}{h} e^{\frac{1}{2}\pi n} \mid \Gamma(k+1-in) \mid \left(\frac{n'}{n}\right)^{k+\frac{1}{2}} \frac{2^{k}}{(2k+1)!!\,k} \\ &(\mathrm{E} > 0, \mathrm{E}' \to \infty).\end{aligned}\right\} \quad . \quad . \quad . \quad (4.81)$$

When E is negative, the expressions (4.56) for I′ and J′ reduce to (4.80) when $n' \to 0$. Thus by (3.44), (3.54),

$$\left.\begin{aligned}(\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}', k+1) &\sim i\alpha \sqrt{\left(\frac{2\pi m}{hn^3} \frac{(n+k!}{(n-k-1)!}\right)} \left(\frac{n'}{n}\right)^{k+\frac{1}{2}} \frac{2^{k+1}}{(2k+1)!!} \\ (\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}', k-1) &\sim i\alpha \sqrt{\left(\frac{2\pi m}{hn^3} \frac{(n+k)!}{(n-k-1)!}\right)} \left(\frac{n'}{n}\right)^{k+\frac{1}{2}} \frac{2^{k}}{(2k+1)!!\,k} \\ &(\mathrm{E} < 0, \mathrm{E}' \to \infty).\end{aligned}\right\} \quad . \quad (4.82)$$

When $\mathrm{E} \to +0$, as well as $\mathrm{E}' \to \infty$, we have from (4.81) or (4.54)

$$\left.\begin{aligned}(+0, k \mid 1/r^2 \mid \mathrm{E}', k+1) &\sim \frac{4\pi m}{h} \sqrt{(2\pi)}\, n'^{\,k+\frac{1}{2}} \frac{2^{k+1}}{(2k+1)!!} \\ (+0, k \mid 1/r^2 \mid \mathrm{E}', k-1) &\sim \frac{4\pi m}{h} \sqrt{(2\pi)}\, n'^{\,k+\frac{1}{2}} \frac{2^{k}}{(2k+1)!!\,k}\end{aligned}\right\} (\mathrm{E}' \to \infty), \quad (4.83)$$

and when $\mathrm{E} \to -0$, (4.82) agrees with (4.57).

4.9. Finally, let E and E′ be nearly equal, so that $\lambda \to 1$. We shall include cases in which k is large, so that $k(\lambda - 1)$ is finite or large; but in these cases we shall assume $k \gg n$. In (3.27), (3.35), put

$$\lambda = 1 + \varepsilon, \qquad u = \frac{\varepsilon}{x} + 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.90)$$

Then

$$\left.\begin{aligned}\mathrm{I} &= -2(2+\varepsilon)\,\varepsilon^{in'\varepsilon} \int_0^1 \frac{(1-x)^{k-in'}(\varepsilon + \{2+\varepsilon\}x)^{k+in'}}{(\varepsilon+2x)^{k+2+in}} \\ &\qquad \times [(k+1)\varepsilon + (k+1-in)x](\varepsilon + x)\,dx \\ \mathrm{J} &= \varepsilon^{in'\varepsilon-1} \int_0^1 \frac{(1-x)^{-1-in'}(\varepsilon + \{2+\varepsilon\}x)^{k-1+in'}}{(\varepsilon+2x)^{k+1+in}} x^2\,dx\end{aligned}\right\} \quad . \quad . \quad . \quad (4.91)$$

If k and n remain finite as $\varepsilon \to 0$, we can proceed to the limit under the integral signs. The convergence of the integrand of I or J is not uniform near $x = 0$; but the integrand is bounded, so that we can integrate from $\eta\ (> 0)$ to 1 and then make $\eta \to 0$. Thus

$$\left.\begin{aligned}\mathrm{I} &\sim -4\int_0^1 (1-x)^{k-in'}\,\tfrac{1}{4}(k+1-in')\,dx = -1 \\ \mathrm{J} &\sim \frac{1}{\varepsilon}\int_0^1 (1-x)^{k-1-in'}\,\tfrac{1}{4}\,dx = \frac{1}{4\varepsilon(k-in')}\end{aligned}\right\} (k, n, \ll 1/\varepsilon). \quad (4.92)$$

If $k \to \infty$ as $\varepsilon \to 0$, we put $x = \varepsilon y$. Then

$$\left.\begin{aligned} \mathrm{I} &= -2(2+\varepsilon)\,\varepsilon \int_0^{1/\varepsilon} \frac{(1-\varepsilon y)^{k-in'}(1+\{2+\varepsilon\}\,y)^{k+in'}}{(1+2y)^{k+2+in}} \\ &\qquad [k+1+(k+1-in)\,y]\,(1+y)\,dy \\ \mathrm{J} &= \int_0^{1/\varepsilon} \frac{(1-\varepsilon y)^{k-1-in'}(1+\{2+\varepsilon\}\,y)^{k-1+in'}}{(1+2y)^{k+1+in}}\,y^2\,dy \end{aligned}\right\} \quad . . . \quad (4.92)$$

Now

$$\frac{(1-\varepsilon y)^k(1+\{2+\varepsilon\}\,y)^k}{(1+2y)^k} = \left[1-\frac{\varepsilon(2+\varepsilon)\,y^2}{1+2y}\right]^k,$$

$$\sim \exp\frac{-2\varepsilon k y^2}{1+2y} \quad . . . . \quad (4.94)$$

wherever it is not negligible. Also

$$(1-\varepsilon y)^{-in'}(1+\{2+\varepsilon\}\,y)^{in'}(1+2y)^{-in} = \left[1-\frac{2\varepsilon y+2\varepsilon y^2}{1+\{2+\varepsilon\}\,y}\right]^{-in'}(1+2y)^{-in'\varepsilon}. \quad (4.95)$$

So long as $k \gg n'$, (4.95) may be replaced by 1, even when n' is large; for whenever (4.94) is not negligible $\dfrac{2\varepsilon n' y^2}{1+2y} \ll 1$, and (4.95) reduces to

$$(1-2\varepsilon y)^{-in'}(1+2y)^{in'\varepsilon} \sim 1.$$

Thus

$$\left.\begin{aligned} \mathrm{I} &\sim -4\varepsilon k \int_0^{1/\varepsilon} e^{-\frac{2\varepsilon k y^2}{1+2y}}\left(\frac{1+y}{1+2y}\right)^2 dy \\ \mathrm{J} &\sim \int_0^{1/\varepsilon} e^{-\frac{2\varepsilon k y^2}{1+2y}}\left(\frac{y}{1+2y}\right)^2 dy \end{aligned}\right\} \quad . . . . \quad (4.96)$$

Putting $t = 2\varepsilon k y^2/(1+2y)$, we have

$$\left.\begin{aligned} \mathrm{I} &\sim -\int_0^\infty e^{-t}\sqrt{\left(\frac{t+2\varepsilon k}{t}\right)}\,dt \\ \mathrm{J} &\sim \frac{1}{4\varepsilon k}\int_0^\infty e^{-t}\sqrt{\left(\frac{t}{t+2\varepsilon k}\right)}\,dt \end{aligned}\right\} \quad (k \gg n'). \quad . . . . . . \quad (4.97)$$

When εk is small, these reduce to (4.92). When εk is large, they give

$$\mathrm{I} \sim -\sqrt{(2\pi\varepsilon k)}, \quad \mathrm{J} \sim \frac{\sqrt{\pi}}{8\sqrt{(2\varepsilon^3 k^3)}} \quad (k \gg 1/\varepsilon, n'). \quad . . . . . . \quad (4.98)$$

Substituting from (4.92) in (3.28), (3.36), we have

$$\left.\begin{aligned} (\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}+0, k+1) &= \frac{4\pi m}{h\sqrt{\{(k+1)^2+n^2\}}} \\ (\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E}+0, k-1) &= \frac{4\pi m}{h\sqrt{(k^2+n^2)}} \end{aligned}\right\} \quad . . . . \quad (4.99)$$

Since $(\mathrm{E}, k \mid 1/r^2 \mid \mathrm{E} - 0,\ k + 1) = (\mathrm{E} - 0, k + 1 \mid 1/r^2 \mid \mathrm{E}, k)^*$, (4.99) shows that the matrix-element, considered as a function of E and E′, is continuous at E = E′. When k is large, so that $(\lambda - 1)\, k$ is not small, we have to multiply the expressions in (4.99) by

$$\lambda^{-k}(-\mathrm{I}), \qquad \lambda^{-k}\, 4k\,(\lambda - 1)\,\mathrm{J}, \qquad (4.9\mathrm{T})$$

where I and J are given by (4.97).

§ 5. *Coefficients of Absorption.*

5.0. We now consider the absorption of energy by an electron in transitions from a bound state to a free state, or from one free state to another of greater energy. Let the wave-function be initially one of the wave-functions discussed in § 2, with the suitable time-factor

$$\psi_0 = \psi\,(\mathrm{E}, k, u)\, e^{-2\pi i \mathrm{E}t/h}. \qquad (5.00)$$

If $\mathrm{E} < 0$ this represents a single bound electron; if $\mathrm{E} > 0$ it represents free electrons encountering the central charge at the rate of one per unit time. Let us introduce radiation whose vector potential near the origin is

$$\mathbf{j}'\mathrm{A} \cos 2\pi\nu t \quad (\mathbf{j}'^2 = 1\ ;\ \mathrm{E} + h\nu > 0), \qquad (5.01)$$

where $\mathbf{j}'$ is a unit vector, and the frequency is sufficiently high to free the electron if it is bound. A free electron cannot absorb radiation except in an encounter with another charge, and it is here assumed that the space around the central charge in which this process is effective is small compared with the wave-length of the radiation. Otherwise, (5.01) should vary with position, and give rise to terms involving quadripole* and higher moments in subsequent formulæ. The interaction energy of the radiation and the electron is

$$\frac{e\mathrm{A}}{c}(\mathbf{j}' \cdot \dot{\mathbf{r}}) \cos 2\pi\nu t, \qquad (5.02)$$

where $\dot{\mathbf{r}}$ is the electron's velocity. The wave-equation is

$$\left[\frac{ih}{2\pi}\frac{\partial}{\partial t} - \mathrm{H}\right]\psi = \frac{e\mathrm{A}}{c}(\mathbf{j}' \cdot \dot{\mathbf{r}}) \cos 2\pi\nu t\ \psi, \qquad (5.03)$$

where H is the original Hamiltonian operator, so that the functions (5.00) satisfy (5.03) when the perturbation on the right vanishes. If the radiation is weak, ψ changes slowly from its original form, and for small t we may replace it by ψ_0 on the right of (5.03).

* See Gaunt and McCrea, 'Proc. Camb. Phil. Soc.,' vol. 23, p. 930 (1927).

At time t we may expand ψ in a series and integral of the wave-functions (2.01), getting

$$\psi = \sum_{\substack{E'<0 \\ k', u'}} a(E', k', u')\, \psi(E', k', u')\, e^{-2\pi i E't/h} + \sum_{k', u'} \int_0^\infty a(E', k', u')\, \psi(E', k', u')\, e^{-2\pi i E't/h}\, dE'/h \quad . \quad . \quad (5.04)$$

where the coefficients a depend on t. Substitution in (5.03) gives

$$\sum_{\substack{E'<0 \\ k', u'}} \dot{a}(E', k', u')\, \psi(E', k', u')\, e^{-2\pi i E't/h} + \sum_{k', u'} \int_0^\infty \dot{a}(E', k', u')\, \psi(E', k', u')\, e^{-2\pi i E't/h}\, dE'/h$$
$$= \frac{2\pi e A}{ihc} (\mathbf{j}' . \dot{\mathbf{r}}) \cos 2\pi \nu t \quad \psi(E, k, u)\, e^{-2\pi i Et/h}. \quad . \quad . \quad . \quad (5.05)$$

Comparison of (5.05) with the expansion theorem, similar to (2.44), in three dimensions shows that

$$\dot{a}(E', k', u') = \frac{2\pi e A}{ihc} (E', k', u' \mid \mathbf{j}' . \dot{\mathbf{r}} \mid E, k, u) \cos 2\pi \nu t\, e^{2\pi i (E'-E) t/h} \quad . \quad . \quad (5.06)$$

Put

$$E' - E = h\nu', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.07)$$

and integrate (5.06), the a's being initially zero except a (E, k, u). We obtain

$$a(E', k', u') = \frac{\pi e A}{ihc} (E', k', u' \mid \mathbf{j}' . \dot{\mathbf{r}} \mid E, k, u) \left[\frac{e^{2\pi i (\nu'+\nu) t} - 1}{2\pi i (\nu' + \nu)} + \frac{e^{2\pi i (\nu'-\nu) t} - 1}{2\pi i (\nu' - \nu)} \right]. \quad (5.08)$$

The first term in (5.08) is important when $\nu' \sim -\nu$, *i.e.*, $E' \sim E - h\nu$, and concerns the stimulated emission. The second term gives the absorption, being important when $E' \sim E + h\nu \ (> 0)$. The number of electrons that have been transferred to energies in this neighbourhood is

$$\sum_{k', u'} \int | a(E', k', u') |^2\, dE'/h \qquad (E' \sim E + h\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.09)$$

We may omit the first term of (5.08), and take the matrix-element outside the integral. Writing x for $2\pi(\nu' - \nu) t$, we find that it makes little difference if we let x range from $-\infty$ to ∞. Then the number of absorptions is

$$\sum_{k', u'} \left(\frac{\pi e A}{hc}\right)^2 |(E', k', u' \mid \mathbf{j}' . \dot{\mathbf{r}} \mid E, k, u)|^2 \int_{-\infty}^{\infty} \frac{2(1 - \cos x)}{x^2} \frac{t}{2\pi} dx,$$
$$= \sum_{k', u'} \left(\frac{\pi e A}{hc}\right)^2 |(E, k, u \mid \mathbf{j}' . \dot{\mathbf{r}} \mid E', k', u')|^2\, t \qquad (E' = E + h\nu). \quad (5.10)$$

We multiply (5.10) by $h\nu$ and divide by t, to obtain the rate of absorption of energy. We also divide by the intensity of the radiation, which is $\pi \nu^2 A^2 / 2c$. We express the matrix-element of the velocity in terms of that of the acceleration by the rule

$$(E, k, u\, |\ddot{\mathbf{r}}|\, E', k', u') = 2\pi i \nu\, (E, k, u\, |\dot{\mathbf{r}}|\, E', k', u'), \quad . \quad . \quad . \quad . \quad . \quad (5.11)$$

and use the inverse square law

$$m\ddot{\mathbf{r}} = -\mathrm{Z}e^2\mathbf{j}/r^2, \quad \ldots \ldots \ldots \quad (5.12)$$

where $\mathbf{j}$ is the unit vector introduced at the beginning of § 3. Thus the rate of absorption of energy from a beam of unit intensity is

$$a_0(\mathrm{E}, k, u\,;\ \nu) = \frac{2\pi e^2}{hc\nu}\left(\frac{\mathrm{Z}e^2}{2\pi m\nu}\right)^2 \sum_{k',u'} \left|\left(\mathrm{E}, k, u\left|\frac{(\mathbf{j}\,.\,\mathbf{j}')}{r^2}\right|\mathrm{E}', k', u'\right)\right|^2,$$

$$= \frac{\mathrm{Z}^2e^6}{2\pi hcm^2\nu^3} \sum_{k',u'} |(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k')|^2\, |(k, u\,|\mathbf{j}|\,k', u')\,.\,\mathbf{j}'|^2. \quad \ldots \quad (5.13)$$

The two possible values of k' are $k \pm 1$. If the direction of the radiation is random, we average over all directions $\mathbf{j}'$. The result is independent of u.

$$a_0(\mathrm{E}, k\,;\ \nu) = \frac{\mathrm{Z}^2e^6}{2\pi hcm^2\nu^3} \sum_{k',u'} |(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k')|^2\, \tfrac{1}{3}\,(k, u\,|\mathbf{j}|\,k', u')\,(k, u\,|\mathbf{j}|\,k', u')^*,$$

$$= \frac{\mathrm{Z}^2e^6}{6\pi hcm^2\nu^3}\left\{\frac{k+1}{2k+1}\,(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k+1)|^2 + \frac{k}{2k+1}\,|(\mathrm{E}', k\,|1/r^2|\,\mathrm{E}', k-1)|^2\right\}, \quad \ldots \quad (5.14)$$

by (3.13), (3.15). Bearing in mind the interpretation of the initial wave-function (5.00), we see that a_0 multiplied by the intensity is the energy absorbed per electron per unit time when $\mathrm{E} < 0$, or the energy absorbed per encounter when $\mathrm{E} > 0$.

5.2. In (5.14) both the energy and the angular momentum of the electron are supposed to be known in the initial state. We usually require an average for various angular momenta. For a bound electron, k ranges from 0 to $n - 1$, and the number of states with given k is $2\,(2k + 1)$ (including the spin factor 2). We have to multiply (5.14) by this weight, sum for k, and divide by $2n^2$, the total weight.

$$a_0(\mathrm{E}, \nu) = \frac{\mathrm{Z}^2e^6}{6\pi hcm^2\nu^3}\,\frac{1}{n^2}\sum_{k=0}^{n-1}\{(k+1)\,|(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k+1)|^2 + k\,|(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k-1)|^2\}$$
$$(\mathrm{E} < 0), \quad \ldots \quad (5.20)$$

For free electrons, we shall define $a_0(\mathrm{E}, \nu)$ to be the rate of absorption of energy from radiation of frequency ν and unit intensity by electrons of energy E and unit mean density. Again, the weight by which (5.14) must be multiplied is proportional to $2k + 1$, but there is another factor to be determined. Classically, if v is the velocity of the electrons at infinity, and p is the perpendicular from the central charge to the asymptote of one of their orbits, then the number of encounters per unit time for which p lies in the range dp is

$$v\,.\,2\pi p\,dp. \quad \ldots \ldots \ldots \quad (5.21)$$

Also the angular momentum is

$$mpv \sim kh/2\pi. \quad \ldots \ldots \ldots \quad (5.22)$$

Using (5.22) in (5.21) and putting $dk = 1$, we see that the frequency of encounters with given k is

$$\frac{h^2}{4\pi m^2 v}(2k+1), \quad \text{(5.23)}$$

since $2k+1$ is known to be a factor. We multiply (5.14) by (5.23) and sum, getting

$$a_0(\mathrm{E}, \nu) = \frac{Z^2 e^6 h}{24\pi^2 c m^4 v \nu^3} \sum_{k=0}^{\infty} \{(k+1)\,|(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k+1)|^2 + k\,|(\mathrm{E}, k\,|1/r^2|\,\mathrm{E}', k-1)|^2\}$$
$$(\mathrm{E} > 0). \quad \text{(5.24)}$$

The derivation of (5.23) given above is a simple application of the correspondence principle. The following method is pure quantum theory. A uniform stream of electrons with the given energy is represented by

$$\psi = e^{i\gamma r \cos\theta}, \quad \text{(5.25)}$$

where, as before,

$$\gamma = \frac{2\pi}{h}\sqrt{(2m\mathrm{E})} = \frac{2\pi m v}{h}. \quad \text{(5.26)}$$

We can expand (5.25) in the form*

$$\psi = \sqrt{\left(\frac{2\pi}{\gamma r}\right)} \sum_{k=0}^{\infty} i^k \frac{2k+1}{2} \mathrm{J}_{k+\frac{1}{2}}(\gamma r)\, \mathrm{P}_k(\cos\theta). \quad \text{(5.27)}$$

Each term of this expansion corresponds to electrons with a given angular momentum about the origin. The normalising factor for $\mathrm{P}_k(\cos\theta)$ is $\sqrt{\{(2k+1)/3\pi\}}$. Also as $r \to \infty$

$$\sqrt{\left(\frac{2\pi}{\gamma r}\right)} \mathrm{J}_{k+\frac{1}{2}}(\gamma r) \sim \frac{2}{\gamma r}\sin(\gamma r - \tfrac{1}{2}k\pi). \quad \text{(5.28)}$$

This may be compared with (2.47). The argument of § 2 shows that the normalising factor by which (5.28) must be multiplied in order to represent one encounter per unit time is $\sqrt{(2\pi m\gamma/h)}$. After removal of the two normalising factors, the coefficient of the k wave-function in (5.27) is $i^k\sqrt{\{(2k+1)\,h/2m\gamma\}}$. The square of the modulus of this coefficient is the frequency of encounters with given k, and reduces to (5.23) in virtue of (5.26).

5.3. We now make use of the various approximations of § 4.

I. $\mathrm{E} \to 0$, $\nu \to 0$. The ratio of E to $h\nu$ is supposed to be finite, so that that of E to E′ is finite and not too near 1. Then for $k = \mathrm{O}(n^{\frac{1}{3}})$ we may use (4.23) and (4.43) in (5.14).

$$a_0(\mathrm{E}, k;\ \nu) \sim \frac{32\pi^3 Z^2 e^6}{3h^3 k^2 c\nu^3} a^6(\mathrm{F}^2 + \mathrm{G}^2) \quad (\mathrm{E} > 0), \quad \text{(5.30)}$$

$$a_0(\mathrm{E}, k;\ \nu) \sim \frac{2\pi Z^2 e^6 \alpha^2}{3h^2 k^2 c m \nu^3 n^3} a^6(\mathrm{F}^2 + \mathrm{G}^2) \quad (\mathrm{E} < 0). \quad \text{(5.31)}$$

* Watson, "Bessel Functions," p. 128.

Small and very large values of k, to which these formulæ do not apply, make no appreciable contribution to a_0 (E, ν). The derivation of a_0 (E, ν) will be discussed in § 6.

II. $E \rightarrow 0$, with ν finite, (4.54) and (4.57) are relevant. We use

$$\Gamma(1 + in')\,\Gamma(1 - in') = in'\,\Gamma(in')\,\Gamma(1 - in') = n'\pi/\sinh n'\pi,$$

and find from (5.24)

$$a_0(E, \nu) \sim \frac{4\pi Z^2 e^6}{3hcm^2 v\nu^3}\,\frac{\pi n'}{e^{\pi n'}\sinh \pi n'}\sum_{k=0}^{\infty} n'^{2k+1}\left\{\frac{(1+n'^2)\ldots(\{k+1\}^2+n'^2)\,2^{4k+2}}{(2k+1)!^2\,(k+1)}P^2 + \frac{(1+n'^2)\ldots(\{k-1\}^2+n'^2)\,2^{4k-2}}{(2k+1)!^2\,k}Q^2\right\} \quad (E \rightarrow +0), \quad . \quad . \quad . \tag{5.32}$$

and from (5.20)

$$a_0(E, \nu) \sim \frac{\alpha^2 Z^2 e^6}{3h^2 cm\nu^3 n^5}\,\frac{\pi n'}{e^{\pi n'}\sinh \pi n'}\sum_{k=0}^{n-1} n'^{2k+1}\left\{\frac{(1+n'^2)\ldots(\{k+1\}^2+n'^2)^2\,2^{4k+2}}{(2k+1)!^2\,(k+1)}P^2 + \frac{(1+n'^2)\ldots(\{k-1\}^2+n'^2)\,2^{4k-2}}{(2k+1)!^2\,k}Q^2\right\} \quad (E \rightarrow -0), \quad . \quad . \quad . \tag{5.33}$$

In each series, the second term of the bracket is to be omitted when $k = 0$.

III. $E + h\nu = 0$. We use (4.6E) and (5.20),

$$a_0(E, \nu) = \frac{32\pi Z^2 e^6 \alpha^2}{3h^2 cm\nu^3 n^2}\sum_{k=0}^{n-1}\frac{(n+k)!}{(n-k-1)!}\left\{(k+1)(4n)^{2k}P'^2 + \frac{4}{k}(4n)^{2k-4}Q'^2\right\}, \tag{5.34}$$

where Q′ is to be omitted when $k = 0$.

IV. $\nu \rightarrow \infty$. We use (4.81) and (4.82),

$$a_0(E, \nu) \sim \frac{2Z^2 e^6}{3hcm^2\,v\nu^3}\,\frac{\pi n e^{\pi n}}{\sinh \pi n}\left\{\sum_{k=1}^{\infty}\frac{(1+n^2)(4+n^2)\ldots(k^2+n^2)\,2^{2k}}{(2k-1)!!^2\,k}\left(\frac{n'}{n}\right)^{2k+1} + 4\,\frac{n'}{n}\right\} \tag{5.35}$$

$$\sim \frac{16\pi Z^2 e^6}{3hcm^2 v\,\nu^3}\,\frac{n'}{1 - e^{-2\pi n}} \qquad (E > 0,\ \nu \rightarrow \infty), \tag{5.36}$$

since n' is small. Also

$$a_0(E, \nu) \sim \frac{Z^2 e^6 \alpha^2}{3h^2 cm\,\nu^3 n^5}\left\{4n' + \sum_{k=1}^{n-1}\frac{(n+k)!}{(n-k-1)!}\,\frac{2^{2k}}{(2k-1)!!^2\,k}\left(\frac{n'}{n}\right)^{2k+1}\right\} \qquad (E < 0,\ \nu \rightarrow \infty), \tag{5.37}$$

of which only the first term is important.

V. $\nu \rightarrow 0$. $E > 0$. By (4.99)

$$a_0(E, \nu) \sim \frac{2Z^2 e^6}{3hcm^2 v\,\nu^3}\;2\sum_{k=1}^{\infty}\frac{k}{k^2 + n^2}. \tag{5.38}$$

The series is divergent. Evidently large values of k are important, and for them (4.99) is not valid. The extra factor λ^{-k} in (4.9T) secures convergence. We may make an estimate of a_0 by taking the series in (5.38) from $k = 1$ to $O(1/\{\lambda - 1\})$. The sum then $\sim \log 1/(\lambda - 1)$, or $\log E/h\nu$. Thus

$$a_0(E, \nu) \sim \frac{4Z^2e^6}{3hcm^2v\,\nu^3} \log E/h\nu \qquad (h\nu \ll E). \tag{5.39}$$

§ 6. *Astrophysical Applications.*

6.0. We have now to compare our results with the formulæ used by Milne and others in calculating the stellar absorption coefficient. These are based upon the work of Kramers (*loc. cit.*), who used classical electromagnetic theory. Assuming that the orbit of the electron was nearly parabolic, that the energy radiated was small compared with the initial energy, and that relativity corrections were negligible, Kramers found that the emission within the frequency range ν, $\nu + d\nu$ in an encounter between an electron and a charge Ze at velocity v and impact radius p was

$$\frac{4\pi^2Z^2e^6}{c^3m^2p^2v^2} P(\gamma_\kappa)\, d\nu, \tag{6.00}$$

where

$$\gamma_\kappa = \frac{2\pi\,\nu m^2p^3v^3}{Z^2e^4}, \tag{6.01}$$

$$P(\gamma_\kappa) = \tfrac{1}{3}\{\phi(\gamma_\kappa)^2 + \psi(\gamma_\kappa)^2\}, \tag{6.02}$$

$$\left.\begin{aligned} \phi(\gamma_\kappa) &= \gamma_\kappa\, i^{4/3}\, 3^{-\frac{1}{2}}\, H^{(1)}_{\frac{1}{3}}(i\gamma_\kappa/3) \\ \psi(\gamma_\kappa) &= \gamma_\kappa\, i^{5/3}\, 3^{-\frac{1}{2}}\, H^{(1)}_{\frac{2}{3}}(i\gamma_\kappa/3) \end{aligned}\right\}, \tag{6.03}$$

and the notation is otherwise that of this paper. Kramers' first assumption may be paralleled in quantum theory by the assumption that the electron's energy is small and positive both before and after the emission, *i.e.*, that E′ and E are small. We therefore compare (6.00) with (5.30). Since (6.00) represents unstimulated emission, the corresponding absorption per unit intensity is to be obtained from it by division by $8\pi h\nu^3 d\nu/c^2$, giving

$$a_0(E, p\,;\,\nu) = \frac{\pi Z^2e^6}{2hcm^2p^2v^2\nu^3} P(\gamma_\kappa). \tag{6.04}$$

Now if in (6.01) we put the angular momentum $mpv = kh/2\pi$, we have

$$\gamma_\kappa = \frac{\nu h^3k^3}{4\pi^2mZ^2e^4} = 6a^3, \tag{6.05}$$

by (4.08); and expressing the Hankel functions of (6.03) in terms of Bessel functions,

$$\begin{aligned} \phi &= 6a^3i^{4/3}\,3^{-\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(2ia^3), \\ &= -\,4a^3\,\{I_{\frac{1}{3}}(2a^3) - I_{-\frac{1}{3}}(2a^3)\} = -\,4a^3G, \end{aligned} \tag{6.06}$$

$$\begin{aligned} \psi &= 6a^3i^{5/3}\,3^{-\frac{1}{2}}H^{(1)}_{\frac{2}{3}}(2ia^3) \\ &= -\,4a^3\,\{I_{\frac{2}{3}}(2a^3) - I_{-\frac{2}{3}}(2a^3)\} = -\,4a^3F, \end{aligned} \tag{6.07}$$

so that by (6.02)

$$P(\gamma_\kappa) = \tfrac{16}{3}\,a^6\,(F^2 + G^2). \tag{6.08}$$

Thus, by (5.30)

$$a_0(\mathrm{E}, k; \nu) = \frac{2\pi^3 Z^2 e^6}{h^3 k^2 c \nu^3} \mathrm{P}(\gamma_\kappa), \quad \ldots \ldots \ldots \quad (6.09)$$

which reduces to (6.04) when we put $hk = 2\pi mpv$.

KRAMERS' formula, then, is verified subject to the conditions governing (5.30). This result may be considered to be an instance of the correspondence principle. KRAMERS' first assumption is expressed by $mpv^2/Ze^2 \ll 1$, which becomes in our notation $k \ll n$. If we ignore the possibility $k = 0$, this involves $n \gg 1$. The conditions to which (5.30) is subject are that both n, $n' \gg 1$ and that $\lambda - 1$, or $h\nu/\mathrm{E}$, is not small; they also exclude extreme values of k, for which a_0 is negligible. It is remarked in § 4.7 that (5.30) covers such cases as $n \gg n' \gg 1$. This is in disagreement with the work of OPPENHEIMER,* who finds an extra factor $\frac{3}{2}\sqrt{3}n'^3$ in this instance. The condition that the frequency must not be too small is a new one, and contrasts with KRAMERS' second assumption. KRAMERS' gives a second formula, based on a nearly linear orbit, for the case $k \gg n$. It involves a function $\mathrm{P}'(\gamma_\kappa)$ which, like P, vanishes rapidly when γ_κ is large. Now (4.07) shows that when k/n is large, a, and so γ_κ, is large unless $\lambda - 1$ is small. Thus KRAMERS' second formula is negligible except when $h\nu \ll \mathrm{E}$, and then it fails to agree with (4.99). In any case, its only use was to estimate a correcting factor g', which has generally been omitted in numerical calculations.

6.1. Since k must be large to make γ_κ finite, we may replace the weighted sum which gives $a_0(\mathrm{E}, \nu)$ by the integral used in classical theory,

$$a_0(\mathrm{E}, \nu) = \int_0^\infty a_0(\mathrm{E}, p; \nu)\, v \,.\, 2\pi p\, dp$$

$$= \frac{\pi^2 Z^2 e^6}{hcm^2 v \nu^3} \int_0^\infty \mathrm{P}(\gamma_\kappa) \frac{d\gamma_\kappa}{3\gamma_\kappa} = \frac{4\pi Z^2 e^6}{3\sqrt{3}hcm^2 v \nu^3} \quad \ldots \ldots \quad (6.10)$$

by a formula of KRAMERS. MILNE† has an extra factor

$$1 - e^{-h\nu/k\mathrm{T}} \quad \ldots \ldots \ldots \ldots \quad (6.11)$$

in (6.04) and (6.10). This factor would be justified if (6.00) included the emission stimulated by black-body radiation at temperature T. This is evidently not so. The real use of (6.11) in MILNE's theory is to make the mean absorption coefficient finite, since the "straight" mean involves the integral

$$\int_0^\infty \frac{\nu^3 a_0(\mathrm{E}, \nu)}{e^{h\nu/k\mathrm{T}} - 1} d\nu, \quad \ldots \ldots \ldots \ldots \quad (6.12)$$

which is divergent without the extra factor. A difficulty of this sort is discussed by EDDINGTON,‡ who seems to argue that KRAMERS' theory includes the stimulated emission since in the absence of incident radiation the electron could not maintain the assumed orbit. The quantum theory, however, does not support this argument. In fact, it

* O II.

† M I, p. 752–3.

‡ EDDINGTON, 'M.N.R.A.S.,' vol. 84, p. 116 (1924).

appears from (5.39) that the correct absorption coefficient gives a more violent divergence than KRAMERS'. It is doubtful, however, whether this large absorption at low frequencies has much importance. The mean required for astrophysical purposes is ROSSELAND'S, which converges without difficulty. In MILNE'S* theory of absorption lines any kind of mean is out of place. We require the absorption coefficient for the continuous background in the immediate neighbourhood of the line, and the appropriate frequency is quite closely defined. The main effect in a star of large absorptions at low frequencies would be to stop the outward flow of radiation of these frequencies. This radiation would form part of an equilibrium distribution at some temperature probably different from that of the material in its neighbourhood. Its amount is in any case small, and since the spontaneous emission is finite, even with (5.39), its effect on the distribution of electrons is not important. The physical meaning of frequent absorptions is not very clear, when we reflect that a low frequency requires a very long wave-train to definite it. Were it not that the Hertzian radiation from a star is presumably less than would otherwise be expected, we might explain away the paradox as a trick of Fourier analysis.

6.2. We return from this digression to consider the absorption by liberation of a bound electron. When the energies before and after the transition are numerically small, (5.31) is appropriate. This expression is formally the same as (5.30), except for an extra factor

$$\frac{\alpha^2 h}{16\pi^2 m n^3} = \frac{4\pi^2 m Z^2 e^4}{h^3 n^3}. \qquad (6.20)$$

When n is large, (6.20) is the frequency-difference, $\Delta\nu$, between successive energy-levels, as may be seen by differentiating (2.06) with respect to n and dividing by h. As before, we may replace the sum by an integral in finding a_0 (E, ν).

$$\begin{aligned} a_0(\mathrm{E}, \nu) &= \frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\, a_0(\mathrm{E}, k; \nu), \\ &\sim \frac{1}{n^2}\int_0^\infty \frac{4\pi^3 Z^2 e^6}{h^3 c \nu^3}\Delta\nu \frac{\mathrm{P}(\gamma_\kappa)\, d\gamma_\kappa}{3\gamma_\kappa}, \\ &= \frac{1}{n^2}\cdot\frac{16\pi^2 Z^2 e^6}{3\sqrt{3} c h^3 \nu^3}\Delta\nu. \end{aligned} \qquad (6.21)$$

This agrees with the formula used by MILNE,† when g' is replaced by unity, since in our case

$$\sigma q'/q = 1/n^2. \qquad (6.22)$$

Here σ is the symmetry number of the atom, q' the weight of the atom after the electron has been ejected, and q the weight of the initial energy-level. Strictly speaking, q' should be the product of the weights of the ion and the free electron; so that with a

* MILNE, 'M.N.R.A.S.,' vol. 89, p. 30 (1928). Referred to as M II.

† M I, p. 754.

spinning electron it is doubled. The weight of the initial energy-level is then $2n^2$ instead of n^2, so that (6.22) remains true. MILNE actually takes for $\Delta\nu$ the difference

$$\Delta\nu = \frac{2\pi^2 m Z^2 e^4}{h^3}\left[\frac{1}{(n-\frac{1}{2})^2} - \frac{1}{(n+\frac{1}{2})^2}\right], \quad \ldots \ldots \quad (6.23)$$

but since we have found (6.21) only for large n, we need not here distinguish between (6.20) and (6.23). We shall return to a discussion of small n later.

6.3. It is now necessary to consider whether the conditions under which the above formulæ are good approximations are realised in astrophysical problems. Looking at the steps leading to the approximation (4.06), we see that the terms neglected are of the order of $1/u^2$ and $1/k$, that is, of order $1/n^{2/3}$ when a is finite. The methods of analysis are ill-suited to determine the accuracy of numerical approximations, and it is always difficult to say how far an asymptotic formula may be trusted. We should probably feel fairly safe if n and n' were as large as 10, and we might hope for the right order of magnitude when they were 3 or 4. It appears, however, that in most cases at least one of them is smaller than this. As two extreme examples of the determination of the coefficient of opacity (ROSSELAND's mean) we shall take the centre and the outer layers of Capella. We shall also take three cases considered by MILNE* in his paper on maxima of absorption lines. When the absorption coefficient varies as $1/\nu^3$, ROSSELAND's mean involves

$$\int_0^\infty \frac{\nu^6}{e^{h\nu/kT}-1}\,d\nu, \quad \ldots \ldots \quad (6.30)$$

and the frequencies of the greatest importance are those for which $h\nu$ is about $6kT$. These are the appropriate frequencies in the first two cases, and in the others we use the frequencies of the lines concerned. We take for E the mean energy $\frac{3}{2}kT$. At the centre of Capella we take $Z = 20$; in the photospheric cases $Z = 1$. Energies are given in electron-volts.

	T.	$h\nu$.	E.	E′.	n.	n'.	
Capella (centre) . .	7·2 . 10⁶	3·7 . 10³	0·9 . 10³	4·6 . 10³	2·4	1·1	
Capella (outer) . . .	5000	2·5	0·6	3·1	4·7	2·1	(6.31)
Sun and Capella . . .	5500	2·3*	0·7	3·0	4·4	2·1	
Sirius (Balmer lines)	10000	1·9 to 3·4	1·3	3·2 to 4·7	3·2	2·1 to 1·7	
Arcturus (H and K) .	5000	3·0	0·6	3·6	4·7	1·9	

* This rough value is half the difference between MILNE's ionisation and excitation potentials.

Looking at these figures—especially those for the centre of Capella—we are not surprised that KRAMERS' formula gives the wrong result. Instead of the opacity of 53 which is required inside Capella, EDDINGTON's calculations give 5. MILNE contrives to multiply this figure by 2 or 3, by a treatment of the bound electrons which will be criticised later.

* M II, pp. 37, 41, 43.

In the outer layers of the Sun, Capella and (perhaps) Sirius, MILNE* finds, from observations on the maxima of lines, absorption coefficients of the same order as that which he calculates from KRAMERS' formula. The calculated coefficient, however, is his mean over the whole spectrum, which is equal to the coefficient for the frequency $\bar{\nu}$, where, (see (6.11), (6.12)),

$$\frac{1}{\bar{\nu}^3} = \int_0^\infty e^{-h\nu/kT}\, d\nu \Big/ \int_0^\infty \frac{\nu^3\, d\nu}{e^{h\nu/kT} - 1} = \frac{h^3}{6\cdot 4\,(kT)^3}. \quad \ldots \ldots \quad (6.32)$$

The frequencies actually concerned are greater than $\bar{\nu}$. The frequencies of the lines observed in the Sun and Capella are not given by MILNE; the rough guess entered above is nearly $6kT/h$. The lines concerned in Sirius range from $2\cdot 3\; kT/h$ to $4\cdot 1\; kT/h$. Since the coefficient of absorption varies as $1/\nu^3$, MILNE's calculation may have to be divided by factors ranging from 2 to 30. Until more precise observational data are to hand the comparison must remain vague.

6.4. Since the conditions under which KRAMERS' formula was verified are not realised in practice, we turn to our other approximations, in order to see what kind of deviations from this formula they would lead us to expect. For this purpose we introduce a correcting factor g, defined by

$$\left.\begin{aligned} a_0\,(\mathrm{E}, \nu) &= \frac{4\pi Z^2 e^6}{3\sqrt{3}\, hcm^2 v \nu^3}\, g \qquad (\mathrm{E} > 0) \\ a_0\,(\mathrm{E}, \nu) &= \frac{Z^2 e^6 \alpha^2}{3\sqrt{3}\, h^2 cm \nu^3 n^5}\, g \qquad (\mathrm{E} < 0) \end{aligned}\right\}, \quad \ldots \ldots \quad (6.40)$$

so that when E, E', $\to 0$, $g \to 1$, by (6.10), (6.21). We have the following results from § 5.3

<u>E = 0.</u> By (5.32), (5.33)

$$g = \frac{\sqrt{3}\,\pi n'}{e^{\pi n'} \sinh \pi n'} \sum_{k=0}^{\infty} n'^{2k+1} \left\{ \frac{(1 + n'^2) \ldots (\overline{k+1}^2 + n'^2)\, 2^{4k+2}}{(2k+1)\,!^2\,(k+1)} \mathrm{P}^2 \right.$$
$$\left. + \frac{(1 + n'^2) \ldots (\overline{k-1}^2 + n'^2)\, 2^{4k-2}}{(2k+1)\,!^2\, k} \mathrm{Q}^2 \right\}. \quad \ldots \quad (6.41)$$

<u>E' = 0.</u> By (5.34)

$$g = 32\pi\sqrt{3}\, n^3 \sum_{k=0}^{n-1} \frac{(n+k)\,!}{(n-k-1)\,!} \left\{ (k+1)(4n)^{2k} \mathrm{P}'^2 + \frac{4}{k}(4n)^{2k-4}\, \mathrm{Q}'^2 \right\}. \quad \ldots \quad (6.42)$$

<u>E' → ∞.</u> By (5.36), (5.37)

$$\left.\begin{aligned} g &\sim 4\sqrt{3}\, n'/(1 - e^{-2\pi n}) \qquad (\mathrm{E} > 0) \\ g &\sim 4\sqrt{3}\, n' \qquad (\mathrm{E} < 0) \end{aligned}\right\}. \quad \ldots \ldots \quad (6.43)$$

<u>ν → 0.</u> By (5.39)

$$g \sim \frac{\sqrt{3}}{\pi} \log \mathrm{E}/h\nu. \quad \ldots \ldots \quad (6.44)$$

* M II, pp. 37, 41.

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Values of g have been found from (6.41) for $n' = \frac{1}{2}$, 1 and 2. Only a few terms of the series are required. The series (4.55) for P and Q were summed numerically. It was necessary to use as many as 15 terms, since the terms increase in modulus at first to as much as 10 times that of the final sum. The results are given in the table (6.45) below. The column headed "$e^{2in'}\,{}_1F_1$" contains $e^{2in'}\,{}_1F_1(k + in'\,;\ 2k\,;\ -4in')$, which occurs in Q. The sixth and seventh columns contain the terms of the series in (6.41), and the eighth column its sum.

n'.	k.	P.	$e^{2in'}\,{}_1F_1$.	Q.	Term in P^2.	Term in Q^2.	Series.	g.
$\frac{1}{2}$	0	1·2	—	—	3·5	—	4·3	1·1
	1	1·1	1·4	3	0·7	0·1		
	2	~1	1·1	< 5	~0·03	< 1/400		
1	0	1·9	—	—	30	—	60	1·2
	1	1·5	3·1	5	20	3		
	2	1·4	1·75	7	5	0·2		
	3	~1·3	1·4	<20	~0·3	< 1/10		
2	0	9	—	—	3200	—	15000	1·1
	1	4·3	29·4	24	4800	500		
	2	3·4	7·6	24	4500	200		
	3	2·3	4·0	17	1100	20		
	4	~2	2·9	<40	~300	< 15		

(6.45)

The last figure in g is uncertain. We can only say that this factor is surprisingly near unity. Calculations for larger values of n' would be more laborious; but we know that as $n' \to \infty$ $g \to 1$, and it seems unlikely that g should differ much from unity for such values. For small n', (6.43) gives $g \backsim 4\sqrt{3}n'$, so that g is then small. Apart, however, from this region of very high frequencies it appears that KRAMERS' formula is a good approximation when E alone $\to 0$. These conclusions are in strong contrast with those of OPPHENHEIMER.* In his notation, g is defined by the equation (*loc. cit.*, p. 726)

$$I^z + I^x = \frac{\pi}{\sqrt{3}}\, Qg. \qquad (6.46)$$

For $E = 0$ (the threshold) and n' (his n) $= 1$, his curves (*loc. cit.*, fig. 3) indicate a value of g of about 8. For $n' = 2$ the curve goes off the diagram, and g must be about 25 or more. In fact, his equations (16) and (19) give for large n': $g \sim 3\sqrt{3n'^3}/2$. It is owing to these large factors near the threshold that OPPENHEIMER finds an opacity approaching the value required in the stars. His argument is obscure, but there seems to be no doubt that he has made a mistake. The approximations for large n and n' have been worked out again on the basis of his general formulæ, and the result is the same as KRAMERS'.† This correction is of crucial importance. It is evident that if his

* O II.

† GAUNT, 'Z. Physik' vol. 59, p. 508 (1930).

curves in fig. 3 are to come down to the unit line near the threshold there is much less chance of a high opacity. OPPENHEIMER's other formulæ agree with ours. Thus his equations (16) and (18) give (in our notation)

$$I^z + I^x \sim 4\pi n' (1 - \pi n') Q \quad (n \to \infty, n' \to 0). \quad \dots \dots \quad (6.47)$$

Since

$$P(n', 0) = 1 + O(n'^2) \quad (n' \to 0), \quad \dots \dots \dots \quad (6.48)$$

(6.47) agrees with (6.41) and (6.46). Again, as $\nu \to 0$ his formulæ (24) reduce to

$$I^z \sim Q, \quad I^x \sim Q \log 1/\delta, \quad (\nu \to 0), \quad \dots \dots \dots \quad (6.49)$$

which agrees with (6.44) since δ is proportional to $h\nu/E$. It is doubtful whether the more complicated forms of his (24) are very valuable. If they were always valid up to three-quarters of the threshold frequency they would be extremely important. However, in deriving the first of formulæ (23) $n/q^{\frac{1}{3}}$, or δ, is neglected in comparison with unity; and the second formula neglects δ^2. Granting (23), I find errors in (24), which are small, however, for small δ. In the limit we have (6.49), but there may be permissible values of δ for which the extra factor $2\pi\delta/(1 - e^{-2\pi\delta})$ amounts to as much as 2. Suppose δ may be as great as $\frac{1}{2}$; then curves 1 and 4 in OPPENHEIMER's fig. 3 are given as far as $\nu/\nu_0 = 0\cdot36$, curve 2 as far as $\nu/\nu_0 = 0\cdot5$, and curves 3 and 5 as far as $\nu/\nu_0 = 0\cdot55$. In these ranges the greatest value of g is about 4.

6.5. An estimate of the contribution of the bound electrons to the absorption is provided by (6.42). The negative energy of the lower levels is generally several times larger than kT, so that it is not often that an electron escapes from such a level with much energy. That is, $E' \sim 0$ is a reasonable assumption when n is not large. The following table shows the results of calculations with (6.42). P' and Q' are calculated by means of the series (4.67), (4.69). None of the series contains more than a few terms.

n.	k.	P'.	Q'.	Term in P'^2.	Term in Q'^2.	Series.	g.
1	0	$\frac{1}{2}e^{-2}$	—	$4\cdot6 . 10^{-3}$	—	$4\cdot6 . 10^{-3}$	0·8
2	0	$-\frac{1}{2}e^{-4}$	—	$1\cdot7 . 10^{-4}$	—	$6\cdot3 . 10^{-4}$	0·9
	1	$\frac{1}{24}e^{-4}$	$-\frac{1}{3}e^{-4}$	$4\cdot5 . 10^{-4}$	$0\cdot1 . 10^{-4}$		
3	0	$\frac{7}{6}e^{-6}$	—	$2\cdot5 . 10^{-5}$	—	$19\cdot3 . 10^{-5}$	0·9
	1	$-\frac{1}{24}e^{-6}$	e^{-6}	$7\cdot4 . 10^{-5}$	$0\cdot4 . 10^{-5}$		
	2	$\frac{1}{720}e^{-6}$	$-\frac{1}{30}e^{-6}$	$8\cdot8 . 10^{-5}$	$0\cdot2 . 10^{-5}$		
4	0	$-\frac{23}{6}e^{-8}$	—	$0\cdot7 . 10^{-5}$	—	$9\cdot2 . 10^{-5}$	1·0
	1	$\frac{3}{40}e^{-8}$	$-\frac{19}{5}e^{-8}$	$1\cdot9 . 10^{-5}$	$0\cdot2 . 10^{-5}$		
	2	$-\frac{1}{720}e^{-8}$	$\frac{7}{90}e^{-8}$	$3\cdot1 . 10^{-5}$	$1\cdot0 . 10^{-5}$		
	3	$\frac{1}{81}e^{-8}$	$\frac{1}{840}e^{-8}$	$2\cdot3 . 10^{-5}$	$0\cdot0 . 10^{-5}$		

(6.50)

Again, g differs little from unity. Thus formula (6.21) is a fair approximation when $E' \sim 0$, even for small values of n. In this formula $\Delta\nu$ is given by (6.20). MILNE,* however, uses for $\Delta\nu$ the difference (6.23). (The return to a differential in his integral (20) is a trivial alteration, since the integrand varies slowly in the range corresponding to small n. The important point is that the upper limit of the integral corresponds to the lowest value of $n - \frac{1}{2}$, not n.) He thus gains an extra factor

$$\frac{n^4}{(n^2 - \frac{1}{4})^2}, \quad \ldots \ldots \ldots \ldots \quad (6.51)$$

which amounts to 16/9 when $n = 1$, and to 256/225 when $n = 2$. Including the factor g from (6.50), we must correct MILNE's values by multiplying the contribution of the K-electrons by $0 \cdot 45$, and that of the L-electrons by $0 \cdot 8$. What he calls the "KRAMERS' factor" is reduced to about $1 \cdot 7$ in each case. The effect upon his calculations for various elements is shown below.

Element.	Fe	Ti	Ca	Ag	
Absorption Coef. (MILNE)......	6·80	14·3	16·3	19·5	(6.52)
Absorption Coef. (corrected) ..	5·13	7·1	7·7	16·9	

It is not suggested that these figures are of much value. The table serves to show that MILNE's appeal to increased absorption by bound electrons to raise the coefficient to something like its astrophysical value must be largely discounted.

§ 7. *Conclusions.*

We cannot profess to have calculated the theoretical value of the absorption coefficient. We have found a number of approximate formulæ which are suitable when some of our parameters are large or small. Table (6.31) shows that the region in which formulæ are required approaches none of these extremes. We can only say that we do not find large deviations from KRAMERS' formula except at very high or very low frequencies. OPPENHEIMER's formulæ (24), with which he hoped to cover most of the ground, are more restricted in their application than he suggests. His large factors near the threshold are due to a mistake.

The difficulty of the large opacity deduced from observations remains. EDDINGTON† has given reasons against the possibility of increasing the contribution of the bound electrons, and table (6.50) supports this view. For free electrons, the important values of the parameters correspond to about $\nu = \frac{3}{4}\nu_0$ in OPPENHEIMER's fig. 3; but when the right-hand half of this diagram is modified to agree with (6.45) at the threshold, it is

* M I, p. 754, equation (8).

† EDDINGTON, "Internal Constitution of the Stars," p. 242 (1926).

difficult to estimate the mean value of g at more than 5 or 6. Now the figures of table (6.52) are due mainly to bound electrons. Multiplying by 6 the part due to free electrons, we find a meagre improvement.

Element.	Fe	Ti	Ca	Ag	
Absorption Coef.	8·88	10·2	10·8	23·3	. . (7.0)

To summarise :

(1) KRAMERS' formula is a good approximation when the energy of the absorbing electron is small.

(2) MILNE's calculation of the absorption by bound electrons is an overestimate.

(3) These results are unfavourable to a high opacity. OPPENHEIMER's high value is due to a mistake.

(4) For very low frequencies the absorption coefficient is even greater than KRAMERS' ; but this fact is not of importance.

(5) For very high frequencies the absorption coefficient is less than KRAMERS'. This result is to be found in an earlier paper by OPPENHEIMER.*

(6) It is suggested that in work on absorption lines the coefficient for the continuous background should be that corresponding to the frequency of the line, and not a mean over the whole spectrum. This affects the variation of the absorption coefficient with temperature. Thus KRAMERS' formula (6.10) yields $k \propto P/T^{9/2}$ (where P is the electron pressure), when both v and ν are averaged suitably to the temperature ; but if ν is kept constant $k \propto P/T^{3/2}$.

* O I, p. 293.

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